Lecture 23

Last time:

$$\frac{d}{dt}\Phi(t,\tau) = A(t)\Phi(t,\tau)$$
$$\Phi(\tau,\tau) = I$$

So the covariance matrix for the state at time t is

$$\begin{aligned}
X(t) &= \overline{\left[\underline{x}(t) - \overline{\underline{x}(t)}\right]\left[\underline{x}(t) - \overline{\underline{x}(t)}\right]^T} \\
&= \overline{\underline{\tilde{x}}(t)\underline{\tilde{x}}(t)^T} \\
&= E\left[\Phi\left(t,t_0\right)\underline{\tilde{x}}(t_0) + \int_{t_0}^{t}\Phi\left(t,\tau_1\right)B(\tau_1)\underline{\tilde{n}}(\tau_1)d\tau_1\right]\left[\underline{\tilde{x}}(t)^T\Phi\left(t,t_0\right)^T + \int_{t_0}^{t}\underline{\tilde{n}}(\tau_2)^T B(\tau_2)^T\Phi\left(t,\tau_2\right)^T d\tau_2\right] \\
&= \Phi\left(t,t_0\right)\overline{\underline{\tilde{x}}(t)\underline{\tilde{x}}(t)^T}\Phi\left(t,t_0\right)^T \\
&\quad + \int_{t_0}^{t}\Phi\left(t,t_0\right)\overline{\underline{\tilde{x}}(t_0)\underline{\tilde{n}}(\tau_2)^T}B(\tau_2)^T\Phi\left(t,\tau_2\right)^T d\tau_2 \\
&\quad + \int_{t_0}^{t}\Phi\left(t,\tau_1\right)B(\tau_1)\overline{\underline{\tilde{n}}(\tau_1)\underline{\tilde{x}}(t_0)^T}\Phi\left(t,t_0\right)^T d\tau_1 \\
&\quad + \int_{t_0}^{t}d\tau_1\int_{t_0}^{t}d\tau_2\Phi\left(t,\tau_1\right)B(\tau_1)\overline{\underline{\tilde{n}}(\tau_1)\underline{\tilde{n}}(\tau_2)^T}B(\tau_2)^T\Phi(t,\tau_2)^T
\end{aligned}$$

The two middle terms are zero:

- For $\tau > t_0$, $\underline{\tilde{n}}(\tau)$ and $\underline{\tilde{x}}(t_0)$ are uncorrelated because $\underline{\tilde{n}}(\tau)$ is white (impulse correlation function)
- For $\tau = t_0$, $\underline{\tilde{n}}(\tau)$ has a finite effect on $\underline{\tilde{x}}(t_0)$ because $\underline{\tilde{n}}(\tau)$ is white. But the integral of a finite quantity over one point is zero.

$$\begin{aligned}
X(t) &= \Phi\left(t,t_0\right)X(t_0)\Phi\left(t,t_0\right)^T + \int_{t_0}^{t}d\tau_1\int_{t_0}^{t}d\tau_2\Phi\left(t,\tau_1\right)B(\tau_1)N(\tau_1)\delta\left(\tau_2-\tau_1\right)B(\tau_2)^T\Phi(t,\tau_2)^T \\
&= \Phi\left(t,t_0\right)X(t_0)\Phi\left(t,t_0\right)^T + \int_{t_0}^{t}\Phi\left(t,\tau\right)B(\tau)N(\tau)B(\tau)^T\Phi(t,\tau)^T d\tau
\end{aligned}$$

This is an integral expression for the state covariance matrix. But we would prefer to have a differential equation. So take the derivative with respect to time.

$$\frac{d}{dt}X(t) = A(t)\Phi\left(t,t_0\right)X(t_0)\Phi\left(t,t_0\right)^T$$
$$+\Phi\left(t,t_0\right)X(t_0)\Phi\left(t,t_0\right)^T A(t)^T$$
$$+\int_{t_0}^{t} A(t)\Phi\left(t,\tau\right)B(\tau)N(\tau)B(\tau)^T\Phi\left(t,\tau\right)^T d\tau$$
$$+\int_{t_0}^{t} \Phi\left(t,\tau\right)B(\tau)N(\tau)B(\tau)^T\Phi\left(t,\tau\right)^T A(t)^T d\tau$$
$$+B(t)N(t)B(t)^T$$
$$\frac{d}{dt}X(t) = A(t)X(t) + X(t)A(t)^T + B(t)N(t)B(t)^T$$

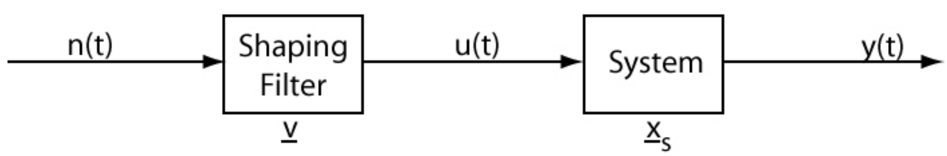
This defines the first and second order statistics of the state.

Initial conditions

Often we wish to compute the time evolution of the statistics of a system which starts from rest at time zero. If the input to this real system is being formed by a shaping filter, then not all elements of X are zero at $t = 0$.

We want to model $x(t)$ as a stationary process.

This situation is equivalent to:



where the white noise input has been applied for all past time. Thus at time zero:
- All elements of $X(0,0)$ which are variances or covariances involving the states of the system are zero.
- All elements of $X(0,0)$ which are variances or covariances involving only states of the shaping filter are at their steady state values for the shaping filter alone driven by the white noise.

$$X = \begin{bmatrix} \text{System states only} & \text{System and filter states} \\ \text{System and filter states} & \text{Filter states only} \end{bmatrix}$$

$$= \begin{bmatrix} \overline{\underline{x}_s \underline{x}_s^T} & \overline{\underline{x}_s \underline{v}^T} \\ \overline{\underline{v}\underline{x}_s^T} & \overline{\underline{v}\underline{v}^T} \end{bmatrix}$$

$$X(0,0) = \begin{bmatrix} 0 & 0 \\ 0 & X_{v,v}(\infty) \end{bmatrix}$$

where

$$x = \begin{bmatrix} \underline{x}_s \\ \underline{v} \end{bmatrix} = \begin{bmatrix} \text{System states} \\ \text{Shaping filter states} \end{bmatrix}$$

With this initialization, $X_{v,v}(t)$ will remain constant – which it should do if we think of $x(t)$ as a member of a stationary process.