Lecture 8

Last time: Multi-dimensional normal distribution

$$f(\underline{x}) = \frac{1}{(2\pi)^{\frac{n}{2}}\sqrt{|M|}} \exp\left[-\frac{1}{2}(\underline{x}-\overline{\underline{x}})^T M^{-1}(\underline{x}-\overline{\underline{x}})\right]$$

If a set of random variables X_i having the multidimensional normal distribution is <u>uncorrelated</u> (the covariance matrix is diagonal), they are <u>independent</u>. The argument of the exponential becomes the sum over i of $\frac{x_i^2}{2}$. Thus, the distribution becomes a product of exponential terms in i.

If $\overline{XY} = 0$

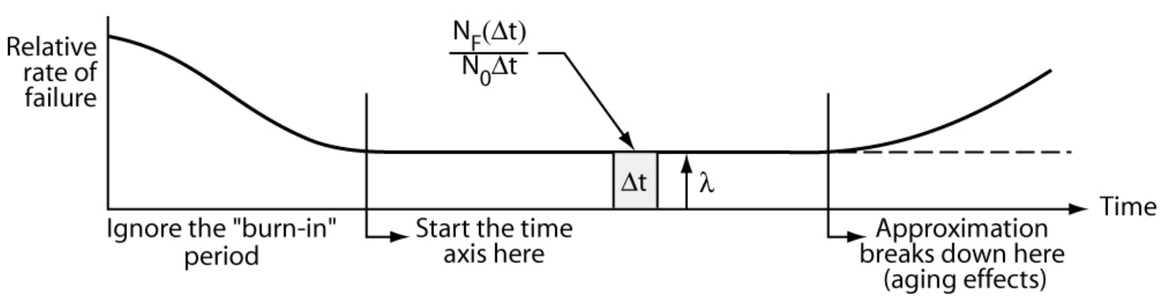
$\overline{X^2Y^3} = \overline{X^2}\,\overline{Y^3} = 0$

The general moment of a multidimensional normal distribution, $E\left[X_1^{r_1}X_2^{r_2}...X_n^{r_n}\right]$, is known.

- Laning and Battin. *Random Processes in Automatic Control.*

<u>The Exponential Distribution</u>

Many components, especially electronic components, display constant percentage failure rates over long intervals.



Relative rate of failure vs time
The familiar "bathtub" curve.

Failure rate = E(relative rate of failure)

Using the random variable T for time to failure, and assuming a constant failure rate λ (independent of time t) implies

$$P(t < T \le t + dt \mid T > t) = \lambda dt$$

This relation alone defines a distribution for time to failure.

$$P(t < T \le t + dt \mid T > t) = \frac{P(t < T \le t + dt \text{ AND } T > t)}{P(T > t)}$$

$$\lambda dt = \frac{f(t)dt}{1 - \int_0^t f(\tau)d\tau}$$

$$f(t) = \lambda - \lambda \int_0^t f(\tau)d\tau \qquad \text{Integral equation for } f(t)$$

$$\frac{df(t)}{dt} = -\lambda f(t) \qquad \text{Differential equation for } f(t)$$

$$f(t) = ce^{-\lambda t}$$

$$\int_0^\infty f(t)dt = -\frac{c}{\lambda} e^{-\lambda t}\Big|_0^\infty = \frac{c}{\lambda}$$

$$= 1$$

$$c = \lambda$$

$$f(t) = \lambda e^{-\lambda t}$$

To find the moments of the distribution, start with the characteristic function.

$$\phi(s) = \int_0^\infty e^{jst} \lambda e^{-\lambda t} dt = \frac{\lambda}{js - \lambda} e^{(js-\lambda)t}\Big|_0^\infty = \frac{\lambda}{\lambda - js}$$

$$\frac{d\phi(s)}{ds} = \frac{\lambda j}{(\lambda - js)^2}$$

$$\frac{d^2\phi(s)}{ds^2} = \frac{2\lambda j^2}{(\lambda - js)^3}$$

$$\overline{T} = \frac{1}{j} \frac{d\phi(s)}{ds}\Big|_{s=0} = \frac{1}{\lambda} = t_m \equiv \text{mean time to failure (MTTF)}$$

$$\overline{T^2} = \frac{1}{j^2} \frac{d^2\phi(s)}{ds^2}\Big|_{s=0} = \frac{2}{\lambda^2}$$

$$\sigma_T^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2} = t_m^2$$

Thus in this case, $\sigma_T = \overline{T}$. The standard deviation is equal to the mean.

$$f(t) = \frac{1}{t_m} e^{-\left(\frac{t}{t_m}\right)}$$

Reliability over lifetime t_l:

$$R(t_l) = P(T > t_l) = \int_t^{\infty} \frac{1}{t_m} e^{-\frac{\tau}{t_m}} d\tau$$

$$= -e^{-\left(\frac{\tau}{t_m}\right)}\Bigg|_{\tau=t_l}^{\infty} = e^{-\left(\frac{t_l}{t_m}\right)}$$

Suppose a system contains n components all of which must operate if the system is to operate (no redundancy), and component failures are considered independent. This is called a *simplex system*.

For the system:

$$\begin{aligned}
R_s(t) &= P(T_s > t) \\
&= P(T_1 > t, T_2 > t, ..., T_n > t) \\
&= P(T_1 > t)P(T_2 > t)...P(T_n > t) \\
&= e^{-\left(\frac{t}{t_{m_1}}\right)} e^{-\left(\frac{t}{t_{m_2}}\right)} ... e^{-\left(\frac{t}{t_{m_n}}\right)} \\
&= e^{-\left(\frac{t}{t_{m_s}}\right)}
\end{aligned}$$

This is the same form for the system as for the individual components. Thus the system also has an exponential distribution of time to failure, with the indicated mean time to failure.

$$\frac{1}{t_{m_s}} = \sum_{i=1}^{n} \frac{1}{t_{m_i}}$$

$$\lambda_s = \sum_{i=1}^{n} \lambda_i$$

If all the components have the same mean time to failure,

$$t_{m_i} = t_{m_c},\ i = 1, 2, ..., n$$

$$\frac{1}{t_{m_s}} = \frac{n}{t_{m_c}}$$

$$t_{m_s} = \frac{1}{n} t_{m_c}$$

Note the importance of part count n to system reliability.

> ***Example: Lifetime system reliability***
>
> Suppose we wish to achieve a 99% probability of successful system operation over a mission lifetime t_l.

$$R_s(t_l) = 0.99$$

$$e^{-\left(\frac{t_l}{t_{m_s}}\right)} = 0.99$$

$$1 - \frac{t_l}{t_{m_s}} = 0.99$$

$$\frac{t_l}{t_{m_s}} = 0.01$$

$$t_{m_s} = 100 t_l$$

To achieve 99% reliability, the MTTF of a component must be at 100 times the mission lifetime.

Now suppose the system contains 1000 components of equal t_{m_c}.

$$\frac{1}{t_{m_s}} = \sum_{i=1}^{1000} \frac{1}{t_{m_i}} = \frac{1000}{t_{m_i}}$$

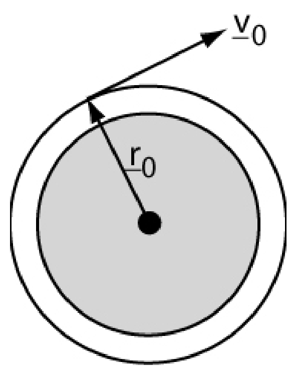
$$t_{m_s} = \frac{1}{1000} t_{m_c} = 100 t_l$$

$$t_{m_c} = 10^5 t_l$$

For a planetary mission where t_l = 2 years, the component MTTF must be 200,000 years. This is why we do not conduct high reliability, high component count missions without redundancy.

Linearized Error Propagation

Example: Orbiting spacecraft



Orbiting spacecraft carrying lander

Given an initial state: $\underline{\hat{x}}_0 = \begin{bmatrix} \underline{r}_0 \\ \underline{v}_0 \end{bmatrix}_{6\times 1}$

Errors are zero mean:

$\underline{e}_0 = \underline{x}_{0_{actual}} - \underline{x}_{0_{est.}}$

Covariance matrix:

$\left[E_0\right]_{6\times 6} = \overline{\underline{e}_0 \underline{e}_0^T}$

Lander deployment is impulsive (assume short burn, negligible change in position):

$$\underline{e}_{r_1} = \underline{e}_{r_0}$$

$$\begin{aligned}\underline{e}_{v_1} &= \underline{v}_{1_{true}} - \underline{v}_{1_{nom}} \\ &= \underline{v}_{0_{true}} + \Delta\underline{v} + \delta\underline{v} - (\underline{v}_{0_{est.}} + \Delta\underline{v}) \\ &= \underline{e}_{v_0} + \delta\underline{v}\end{aligned}$$

$$\begin{aligned}\underline{e}_1 &= \begin{bmatrix} \underline{e}_{r_0} \\ \underline{e}_{v_0} + \delta\underline{v} \end{bmatrix} \\ &= \underline{e}_0 + J\delta\underline{v}, \quad \text{where } J = \begin{bmatrix} 0 \\ I \end{bmatrix}_{6\times 3}\end{aligned}$$

$$\begin{aligned}E_1 &= \overline{\underline{e}_1 \underline{e}_1^T} \\ &= \overline{\left[\underline{e}_0 + J\delta\underline{v}\right]\left[\underline{e}_0^T + \delta\underline{v}^T J^T\right]} \\ &= E_0 + JDJ^T, \quad \text{if } \overline{\underline{e}_0 \delta\underline{v}^T} = \overline{\delta\underline{v}^T \underline{e}_0^T} = 0 \\ &= E_0 + \begin{bmatrix} 0 & 0 \\ 0 & D \end{bmatrix}_{6\times 6}\end{aligned}$$

$D = \overline{\delta\underline{v}\delta\underline{v}^T}$: covariance matrix for velocity correction errors.

Now these errors must be propagated to the surface. A linearized description of error propagation is given by a transition matrix which relates perturbations in position and velocity components at the initial point to perturbations in position and velocity at any later point. For the present purpose, we may be interested only in the position perturbation at the end point.

Direct sensitivity analysis

$$\delta r_{s_i} = \sum_{j=1}^{6} \frac{\partial r_{s_i}}{\partial x_{ij}} \delta x_{ij}, \quad i = 1,2$$

Often this sensitivity matrix can only be evaluated by simulation. If a small δx_{ij} is introduced and the δr_{s_i} noted (i =1,2), the ratios $\frac{\delta r_{s_i}}{\delta x_{ij}}$ are finite-difference measures of the $\frac{\partial r_{s_i}}{\partial x_{ij}}$, which are the values comprising the j^{th} column of S. Thus 6 perturbed trajectories must be calculated and the end state differenced with the nominal end state to define S. <u>Each perturbed trajectory</u> defines <u>one column of S</u>.

$$\underline{e}_s = [S]_{2\times 6}\,\underline{e}_1$$

$$S_{ij} = \frac{\partial r_{s_i}}{\partial x_{ij}} \approx \frac{\Delta r_{s_i}}{\Delta x_{ij}}$$

<u>Linearized analysis</u>

Given the dynamics

$$\dot{\underline{x}} = \underline{f}(\underline{x})$$

$$\dot{\underline{r}} = \underline{v}$$

$$\dot{\underline{v}} = \underline{a} + \underline{g}$$

Consider the errors as a small perturbation around the nominal $\underline{x}(t)$ trajectory.

$$\dot{\underline{x}} = \dot{\underline{x}}_n + \delta\dot{\underline{x}} = \underline{f}(\underline{x}_n + \delta\underline{x})$$

$$\approx \underline{f}(\underline{x}_n) + \frac{d\underline{f}}{d\underline{x}}\delta\underline{x}$$

$$\dot{\underline{x}}_n = \underline{f}(x_n)$$

$$\delta\dot{\underline{x}} = F\delta\underline{x}, \quad \text{where } F_{ij} = \frac{\partial f_i}{\partial x_j}$$

$$\partial\underline{x}(t) = \Phi(t_0,t)\delta\underline{x}(t_0)$$

$$\delta\dot{\underline{x}}(t) = \frac{d\Phi(t_0,t)}{dt}\delta\underline{x}(t_0) = F(t)\delta\underline{x}(t)$$

$$= F(t)\Phi(t_0,t)\delta\underline{x}(t_0)$$

Therefore, for arbitrary $\delta\underline{x}(t_0)$

$$\frac{d\Phi(t_0,t)}{dt} = F(t)\Phi(t_0,t)$$

$$\Phi(t_0,t_0) = I$$