Chapter 3

Phase-plane analysis: introduction

This chapter represents the first chapter in which we will study nonlinear systems characteristics in detail. The set of techniques under the denomination "phase-plane analysis" is especially designed for systems with two states only. This already encompasses a number of significant situations, including the various motions for a rigid mass in one direction (such as a satellite), and many electronic systems. The basic idea behind phase-plane analysis is that trajectories of second-order systems may be plotted in a plane, and therefore easily visualized on a sheet or a computer screen. In this chapter, we develop the basic techniques of phase-plane analysis, that we will later use on specific examples.

3.1 Introduction

3.1.1 State-space equations

In this chapter, we will mostly be concerned with second-order systems defined by the differential equation

$$\ddot{y} = f(y, \dot{y}). \tag{3.1}$$

Usually, such equations arise from the study of Mechanical or Electrical systems, such that the variable y is often the position of a given mass, or the voltage accross a capacitor, and, correspondingly, $\dot{x}$ is the speed or the current intensity going in or out of the same capacitor.

Unless necessary, in order to convert the equation (3.1) to a set of first-order equations, it is always advisable to choose

$$\begin{aligned} x_1 &= y, \\ x_2 &= \dot{y} \end{aligned}$$

as a set of state-space variables, leading to the set of first-order differential equations

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= f(x_1, x_2) \end{aligned} \tag{3.2}$$

Of course, this is not the only available set of variables leading from (3.1) to a set of two first-order equations. For example, defining the two independent variables

$$\begin{aligned} z_1 &= y + \dot{y} \\ z_2 &= y - \dot{y} \end{aligned}$$

would lead one to the set of first-order equations

$$\begin{aligned} \dot{z}_1 &= \dot{y} + f(y, \dot{y}) = (z_1 - z_2)/2 + f((z_1 + z_2)/2, (z_1 - z_2)/2) \\ \dot{z}_2 &= \dot{y} - f(y, \dot{y}) = (z_1 - z_2)/2 - f((z_1 + z_2)/2, (z_1 - z_2)/2). \end{aligned}$$

Apart from its obvious simplicity, we will see later on that the state-space representation (3.2) has interesting graphical properties.

3.1.2 The phase plane

The phase plane is a plane whose horizontal coordinates represent x_1 in the previous set of equations and the vertical coordinates represent x_2. It is the most convenient way to "visualize" a second-order system. For example, Fig. 3.1 shows a number of trajectories which may be followed by a damped mass-spring system satisfying the equation

$$\ddot{y} + 0.4\dot{y} + y = 0.$$

The phase plane coordinates in this case are y and $\dot{y}$, where y is the position of the mass. The goal of this introductory chapter is to tell you a few of the basic properties of trajectories in the phase plane and introduce you to simple methods to draw them "on the back of an envelope", much in the way you have already learned how to draw root-locuses or Nyquist plots.

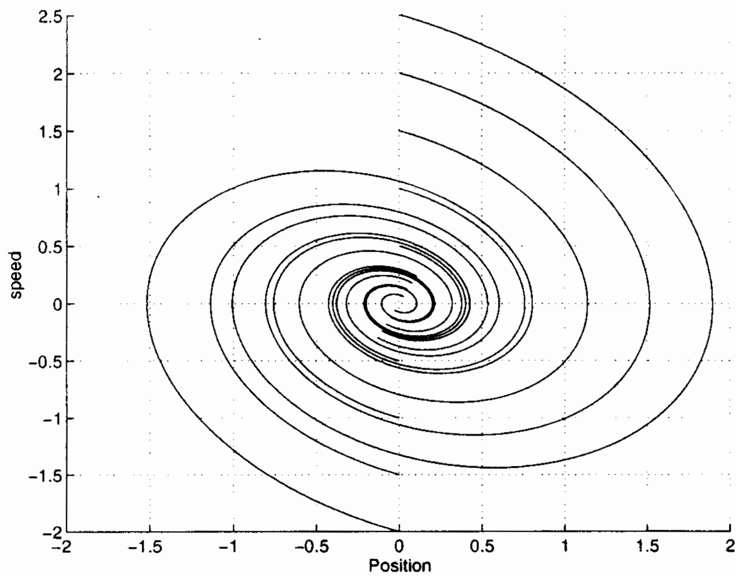

Figure 3.1: Phase-plane trajectories for a mass-spring system.

3.2 General properties

3.2.1 Trajectories as functions of time

Consider the differential equation

$$\frac{d}{dt}x = F(x,t), \quad x(0) = x_0;$$

The first question we ask is: does there exist a solution to this differential equation? In general, you've commonly heard about *the* solution to a differential equation; however, the truth is that there may be *none*, there may be *many*, they may span over a *finite* or *infinite* amount of time.

For example, the scalar differential equation

$$\frac{d}{dt}x = 2\left|x\right|^{1/2}, \quad x(0) = 0$$

admits $x = 0$ as a solution, as well as $x = t^2$. In fact there are many other solutions (you find them in homework).

You may wonder "well, this can't be; there is an error with the modelling"; you will see in homework that some (pathological) systems may be modelled that way

Consider now the scalar differential equation

$$\frac{d}{dt}x = x^2, \quad x(0) = 1.$$

It is a matter of elementary calculus to compute the unique solution to this equation:

$$x(t) = \frac{1}{1-t}$$

which is defined for $t \in [0\ 1)$ only.

In general, these differential equations represent fairly pathological systems (in the second case, an explosion). Most physical systems are "well-behaved", and a large number of them may be characterized via simple conditions: Consider the dynamic system

$$\frac{d}{dt}x = F(x), \tag{3.3}$$

and assume there exists a constant, positive number h such that for any x_1 and x_2,

$$\|F(x_2) - F(x_1)\| \leq h\,\|x_2 - x_1\|$$

where $\|.\|$ is you favourite distance function. (This property is named *Lipschitz continuity.*) Then, for any initial condition $x(0) = x_0$, there exists one and only one solution to (3.3) going through $x(0)$ at $t = 0$ and it is defined for any $t \in \mathbf{R}$. An interesting consequence of this is that trajectories can never cross under these conditions.

3.3 Trajectories in the phase plane

Now, let us see what happens if (3.3) is in fact a second-order system and we decide to plot its trajectories in the phase plane. The previous result tells us that through one point there usually goes one and only one trajectory. While trajectories may be plotted using adequate numerical integration starting from a large number of initial conditions, we now present a set of techniques that will allow you to quickly draw these trajectories "by hand", and hopefully improve your insight on the system under study. In particular, we will present modest tools to plot trajectories (isoclines), to time them and to locate their equilibria.

3.3.1 Isoclines

Consider the second-order system

$$\begin{aligned} \frac{d}{dt}x_1 &= f_1(x_1, x_2), \\ \frac{d}{dt}x_2 &= f_2(x_1, x_2). \end{aligned} \tag{3.4}$$

Then, at each point (x_1, x_2) in the phase plane, the slope of the trajectory going through the point (x_1, x_2) is given by

$$m = \frac{\frac{d}{dt}x_2}{\frac{d}{dt}x_1} = \frac{dx_2}{dx_1} = \frac{f_2(x_1, x_2)}{f_1(x_1, x_2)},$$

which is well defined if $f_1(x_1, x_2) \neq 0$.

An isocline (with slope m) is defined to be the curve satisfying the equation

$$f_2(x_1, x_2) - m f_1(x_1, x_2) = 0.$$

This curve represents the set of points where the slope of the trajectories has the same value m. Unlike trajectories, isoclines are relatively easy to plot and provide a lot of information on the set of all trajectories for a given system.

In the case when canonical coordinates are used for the second-order system

$$\ddot{y} = f(y, \dot{y}),$$

that is $x_1 = y$ and $x_2 = \dot{y}$, then the slope of the trajectory passing through a point (x_1, x_2) is

$$m = \frac{f(x_1, x_2)}{x_2}.$$

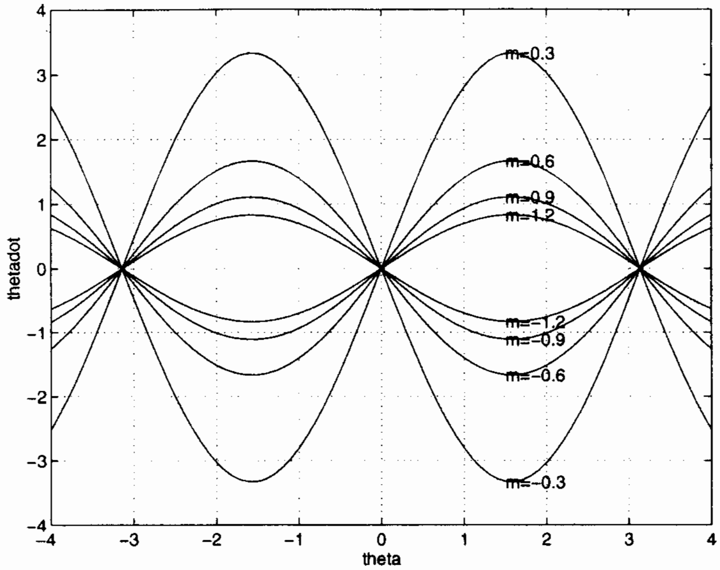

Figure 3.2: Set of isoclines for an undamped pendulum

In Fig. 3.2, we have plotted the isoclines corresponding to an undamped pendulum with equation

$$\ddot{\theta} + \sin\theta = 0,$$

using canonical coordinates.

Trajectory orientation

When using canonical coordinates, it is always good to remember that if speed is positive, then position should go up, if it is negative, position should go down, and if it is zero, then position cannot change. Thus, for x_2 positive (positive speed), the trajectories are all oriented from left to right, for x_2 negative, the trajectories are all oriented from right to left, and when $x_2 = 0$, the trajectories should be vertical, as illustrated in Fig. 3.3 .

3.3.2 Trajectory timing

Phase plane trajectories which are plotted using isoclines do not contain any time information: that is, we don't know how much time is

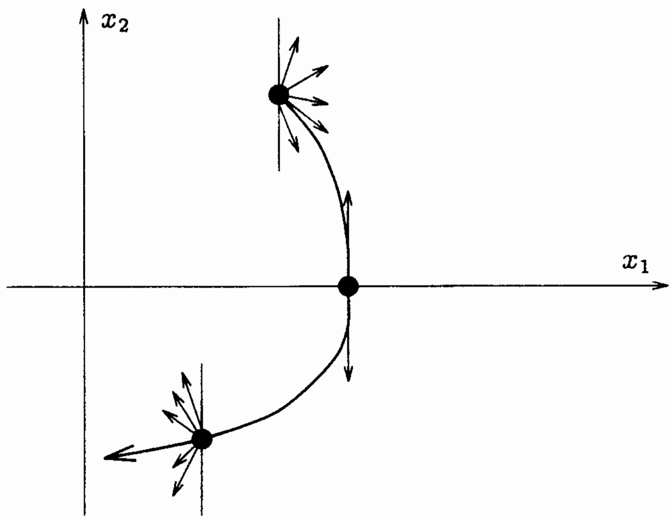

Figure 3.3: Trajectory orientation rules

actually necessary to move from one point on a given trajectory to another on the same trajectory. We now present a few tools to help timing trajectories, in the case of canonical coordinates, only based on a trajectory plot on the phase-plane.

Trajectory speed

We define the trajectory speed as the speed of the state (x_1, x_2) along its trajectory. You must not confuse this speed with the speed of the physical system (*i.e.* $x_2 = \dot{y}$). The horizontal component of the trajectory speed is $dx_1/dt = x_2$. The vertical component of speed is $dx_2/dt = f(x_1, x_2)$. Thus, the complete trajectory speed is $\sqrt{x_2^2 + f(x_1, x_2)^2}$. Geometrical considerations show that this speed is obtained by measuring the distance between the two points M and N, located on the line perpendicular to the trajectory shown in Fig. 3.4. (Provided the axis have the same scale, which is not automatically the case, especially with MATLAB).

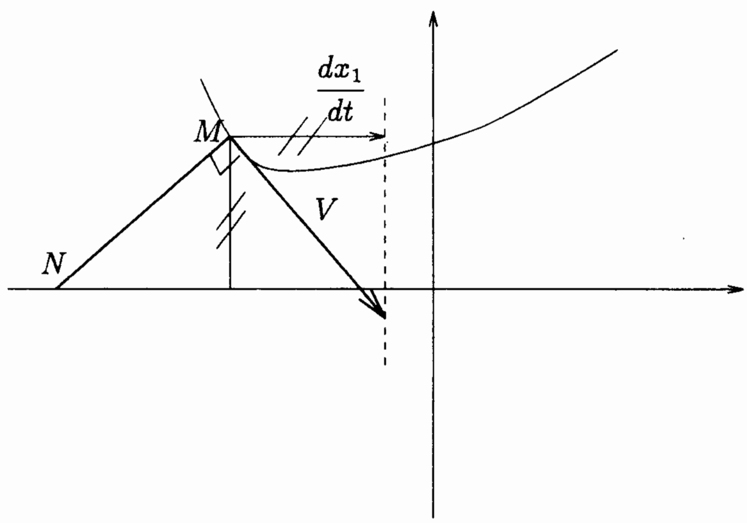

Figure 3.4: Computing the trajectory speed.

Trajectory timing

Once again, using canonical coordinates, we have

$$\frac{d}{dt}x_1 = x_2 \Rightarrow dt = \frac{dx_1}{x_2} \Rightarrow t_{AB} = \int_{x_{1B}}^{x_{1A}} \frac{dx_1}{x_2}.$$

In other terms, in order to compute the time to move from one point to the other, one needs to plot the trajectory $(x_1, 1/x_2)$ and compute the area between this trajectory, the horizontal axis, and the two vertical lines passing through x_{1A} and x_{1B}.

Consider the linear system with equation

$$\ddot{y} + 2\dot{y} + y = 0$$

In Fig. 3.5, we have plotted a typical trajectory and its inverse. As can be seen, the time spent far away from the horizontal axis is considerably less than the one spent near the horizontal axis.

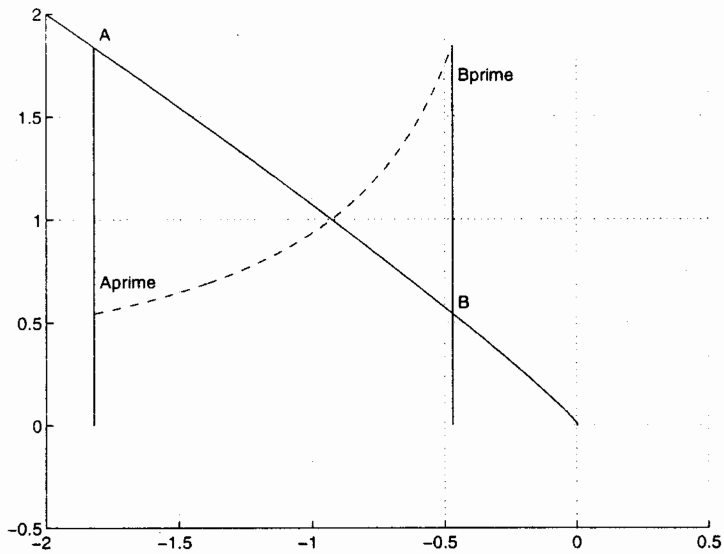

Figure 3.5: A trajectory (plain) and its inverse (dashed) w.r.t. the horizontal axis.

3.3.3 Singular points

Introduction

The determination of singular points represent an essential step in the process of plotting the phase portrait of a nonlinear system: singular points are the points for which the dynamical system

$$\dot{x} = F(x)$$

is at rest, that is, for which $F(x) = 0$. Thus they represent *points of equilibrium* for the system, and they deserve special attention. In particular, it is important to determine the stability status of these points. While in most linear systems, there is only one singular point (what are the exceptions?), for nonlinear systems, there may be many, corresponding to many equilibria. An example of this fact is the *stepper motor*, shown in Fig. 3.6: the rotor and stator have the same number of armatures and magnets, respectively, and it is also the number of possible equilibria this motor can achieve. Let θ be the angular deviation of the stepper motor. A valid set of equations of motion for the

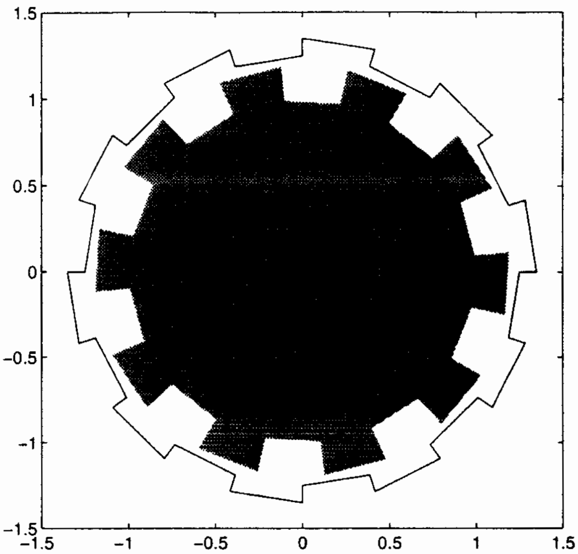

Figure 3.6: rotor and stator for a stepper motor

motor is

$$\ddot{\theta} + 0.1\dot{\theta} + \sin N\theta/2\pi = 0.$$

As can be seen from Fig. 3.6, there can be many equilibria for this system.

Equilibria are very special points in the phase plane: if they are *stable*, then some trajectories in the phase plane will eventually converge towards these points. If they are unstable, trajectories will diverge away from these points: in any case, we expect to encounter trajectory conbcentrations around these points; for example, Fig. 3.1 shows a set of trajectories which all seem to be converging towards the point $(0,0)$.

Stability of singular points

When building a phase portrait for a nonlinear system, it is in general essential to be able to characterize the stability properties of singular points. As we will see now, determining this stability status often goes through the study of stability of the origin for linear systems; indeed, the equations of motion for the nonlinear system

$$\frac{d}{dt}x = F(x)$$

may be expanded around a singular point x_0 as

$$\begin{aligned}\frac{d}{dt}(x - x_0) &= F(x_0) + \Delta F(x_0).(x - x_0) + \|x - x_0\|^2\, \mathcal{O}(1) \\ &= \Delta F(x_0).(x - x_0) + \|x - x_0\|^2\, \mathcal{O}(1),\end{aligned}$$

where $\|.\|$ is your favourite distance measure between x and x_0. Thus, around $x = x_0$, the equation is "almost" like the linear system

$$\frac{d}{dt}(x - x_0) = \Delta F(x_0).(x - x_0).$$

Thus, by studying the behavior of the linear system, one should learn about the behavior of the nonlinear one, at least in the neighborhood of the singular point. This has first been formalized and proved by Lyapunov, and is named *Lyapunov's indirect method for stability*: in the case when $\Delta F(x_0)$ is a matrix whose eigenvalues have nonzero real part, the nature of the stability of the singular point x_0 for the nonlinear system is locally the same as the one of the corresponding linear system. The case when some eigenvalues have zero real part requires more sophisticated tools and we be dealt with later. This basic result leads us to first study the behavior of linear systems near the origin. Its extension to nonlinear systems will then be obvious.

Singular points for linear, second-order systems

The behavior of the second-order system

$$\frac{d}{dt}x = Ax$$

with A a 2×2 matrix is essentially guided by the position of the two eigenvalues of this matrix.

First case; real eigenvalues with opposite sign (saddle point): Let λ_1 and $-\lambda_2$ be these eigenvalues, with $\lambda_1, \lambda_2 > 0$. Then, in an appropriate set of coordinates $(\tilde{x}_1, \tilde{x}_2)$, A may be made diagonal, and the equations of motion now read

$$\begin{aligned}\dot{\tilde{x}}_1 &= \lambda_1 \tilde{x}_1 \\ \dot{\tilde{x}}_2 &= -\lambda_2 \tilde{x}_2,\end{aligned}$$

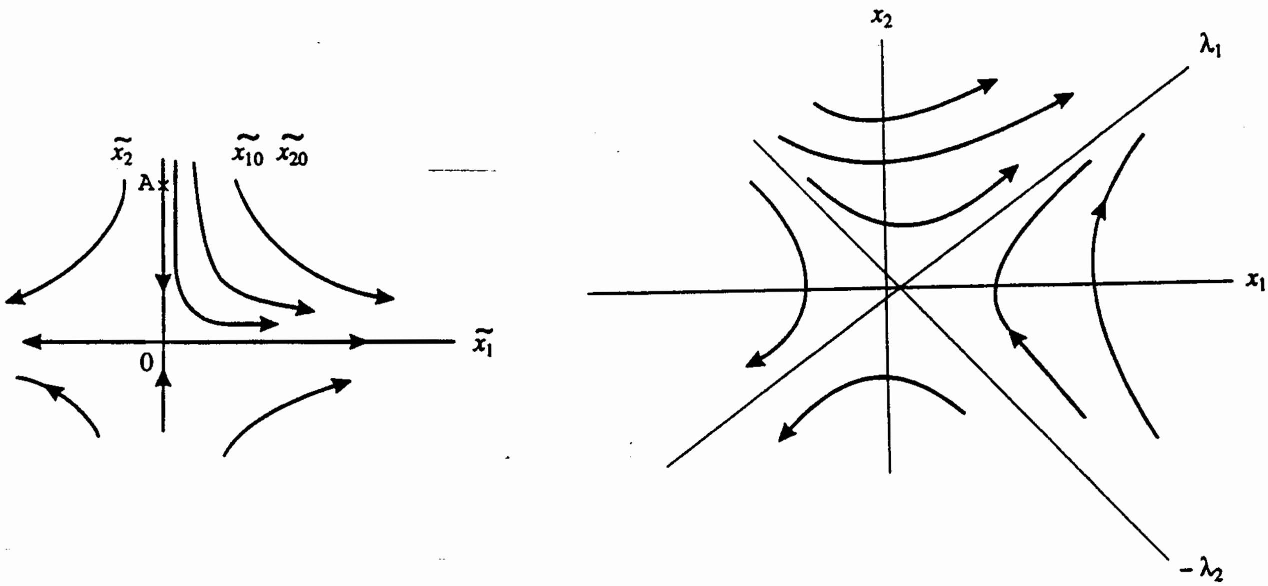

Figure 3.7: "Saddle point". Left: principal coordinates; Right: original coordinates

whose solutions are given by

$$\begin{aligned}\tilde{x}_1 &= \tilde{x}_{10}e^{\lambda_1 t}\\ \tilde{x}_2 &= \tilde{x}_{20}e^{-\lambda_2 t}.\end{aligned}$$

Eliminating time, we obtain the trajectory equations

$$\tilde{x}_2 = \tilde{x}_{20}\left(\frac{\tilde{x}_1}{\tilde{x}_{10}}\right)^{-\frac{\lambda_2}{\lambda_1}} = \frac{K}{\tilde{x}_1^k} \text{ with } k = \frac{\lambda_2}{\lambda_1} > 0.$$

The resulting trajectories are shown in Fig. 3.7. As can be seen, they are made up of hyperbola-like curves, along with the two axes. One of these directions is unstable, whereas the other one is stable. So the singular point is unstable. A physical system exhibiting such a behavior is the inverted pendulum near its unstable equilibrium. In the original basis, we will also find similar behaviors, except that the main axis (the unstable and stable one) will no longer necessarily be orthogonal (see the right picture in Fig. 3.7).

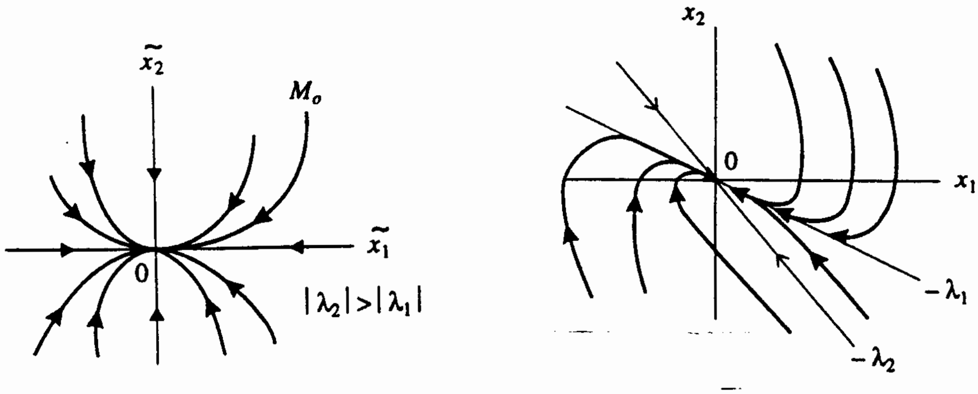

Figure 3.8: Two negative, real eigenvalues. Left: principal coordinates; Right: original coordinates

Second case; real eigenvalues with negative sign: Using the same notation as before, let $-\lambda_1$ and $-\lambda_2$ the eigenvalues of the system, with $\lambda_1 > 0$ and $\lambda_2 > 0$. Then in an appropriate basis, the equations of motion are of the form

$$\dot{\tilde{x}}_1 = -\lambda_1 \tilde{x}_1$$
$$\dot{\tilde{x}}_2 = -\lambda_2 \tilde{x}_2,$$

whose solutions are given by

$$\tilde{x}_1 = \tilde{x}_{10} e^{-\lambda_1 t}$$
$$\tilde{x}_2 = \tilde{x}_{20} e^{-\lambda_2 t}.$$

Eliminating time, we obtain the trajectory equations

$$\tilde{x}_2 = \tilde{x}_{20} \left(\frac{\tilde{x}_1}{\tilde{x}_{10}} \right)^{\frac{\lambda_2}{\lambda_1}} = K \tilde{x}_1^k \text{ with } k = \frac{\lambda_2}{\lambda_1} > 0,$$

So trajectories in the phase plane look like "parabolas". Assume $\lambda_2 > \lambda_1$. Then, the trajectories look like the ones shown in Fig. 3.8. All trajectories are tangent to the horizontal axis, except for the one which is the vertical axis. Such a stable equilibrium is encountered when dealing with overdamped systems.

Third case; real eigenvalues with positive sign: This case is exactly the same as the previous one, if one decides to play the dynamics equations "backward" rather than forward in time. Thus, the trajectories will look exactly the same, except the arrows need to be reversed: the singular point is then unstable.

Fourth case; complex eigenvalues: In this case we know that the eigenvalues of the system are complex, and therefore taking a diagonal representation of the system will lead us to complex solutions, with no obvious physical meaning. We therefore choose to stay in the most natural, canonical basis: A typical system exhibiting oscillatory behavior is a lightly damped mass-spring system, whose equations are

$$\ddot{y} + 2\delta\omega_0\dot{y} + \omega_0^2 y = 0,$$

with $\delta < 1$. Using the canonical coordinates leads us to the equivalent system

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega_0^2 & -2\delta\omega_0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix},$$

the solutions of which are of the form

$$\begin{aligned} x_1 &= A\cos(\omega_0\sqrt{1-\delta^2}t + \phi)e^{-\delta\omega_0 t} \\ x_2 &= -\omega_0\sqrt{1-\delta^2}A\sin(\omega_0\sqrt{1-\delta^2}t + \phi)e^{-\delta\omega_0 t} - \delta\omega_0 x_1. \end{aligned}$$

Let $\psi = \omega_0\sqrt{1-\delta^2}t + \phi$. Then it is possible to write

$$\begin{aligned} x_1 &= B\cos\psi e^{-\frac{\delta}{\sqrt{1-\delta^2}}\psi} \\ \frac{x_2 - \delta\omega_0 x_1}{\omega_0\sqrt{1-\delta^2}} &= B\sin\psi e^{-\frac{\delta}{\sqrt{1-\delta^2}}\psi}. \end{aligned}$$

Let $\tilde{x}_1 = x_1$ and $\tilde{x}_2 = (x_2 - \delta\omega_0 x_1)/(\omega_0\sqrt{1-\delta^2})$. Then, defining $r = \sqrt{\tilde{x}_1^2 + \tilde{x}_2^2}$, the pair (r, ϕ) is a set of polar coordinates that can replace $(\tilde{x}_1, \tilde{x}_2)$. From the previous results, we obtain the relation:

$$r = Ke^{-\frac{\delta}{\sqrt{1-\delta^2}}\psi}.$$

Thus, the corresponding trajectories are logarithmic spirals. With positive damping, these spirals converge towards the origin, which is a stable equilibrium. With negative damping, such spirals move away from the origin, which is unstable. Examples of oscillatory systems with positive damping include classical mass-spring systems. Examples of systems with negative damping include the square section pendulum you'll deal with during the lab.

One example of such logarithmic spirals is given in Fig. 3.1.

Building phase portraits of nonlinear systems (rough)

Summarizing the previous paragraphs, a systematic procedure to build the phase portrait of a nonlinear system is as follows:

- Write the systems equations as a first-order, two degrees-of-freedom system:

$$\begin{aligned}\frac{d}{dt}x_1 &= f_1(x_1, x_2)\\ \frac{d}{dt}x_2 &= f_2(x_1, x_2)\end{aligned}$$

  If possible, write this system in the canonical variables to obtain a form like

$$\begin{aligned}\frac{d}{dt}x_1 &= x_2\\ \frac{d}{dt}x_2 &= f(x_1, x_2).\end{aligned}$$

- Determine the positions of the singular points, solutions to the equations $dx_1/dt = 0$ and $dx_2/dt = 0$. There may be none, one, many, or an infinity of them. When expressed in the canonical coordinates, all singular points have to lie on the horizontal axis. Why?

- Linearize the system around the singular points, to determine the local stability of the corresponding equilibrium. In order to do that, it is necessary that the linearization happens *around the singular point.* In particular, compute the main directions corresponding to these equilibria. If none of the eigenvalues has zero real part, the local behavior of the system is the one of the linearized system around that singular point.

- Complete the diagram by drawing the isoclines and plot "sample" trajectories.

Let us now illustrate this methodology on the system described by the equation

$$\ddot{x} + 4\dot{x} - 2.5x^2 + 5x = 0.$$

- **Choosing the equations of motion**

  Choosing as coordinates the canonical set

$$\begin{aligned} x_1 &= x \\ x_2 &= \dot{x}, \end{aligned}$$

  the corresponding first-order system is given by

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -5x_1 - 4x_2 + 2.5x_1^2 \end{bmatrix}.$$

- **Identify the singular points**

  We then compute the singular points for such a system: they are defined by the equations: $x_2 = 0$, $-5x_1 + 2.5x_1^2 = 0$, thus leading to the two singular points $(0,0)$ and $(2,0)$.

- **Checking stability of the singular points**

  The first singular point is located at the origin, and the equations for the linearized system are obvious:

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -5 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

  The corresponding characteristic polynomial is $s^2 + 4s + 5$, whose roots are $-2 \pm i$. Thus, this singular point is locally stable, and oscillatory with high level of damping.

  The second singular point is at $(2,0)$. Thus it is necessary to linearize the system around that point; defining the translated coordinates $\tilde{x}_1 = x_1 - 2$ and $\tilde{x}_2 = x_2$, we obtain the equivalent set of equations:

$$\begin{aligned} \frac{d}{dt}\tilde{x}_1 &= x_2 \\ \frac{d}{dt}\tilde{x}_2 &= 5\tilde{x}_1 - 4x_2 + 2.5\tilde{x}_1^2. \end{aligned}$$

Linearized around $(0,0)$, this system becomes

$$\frac{d}{dt}\begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 5 & -4 \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix}$$

whose characteristic polynomial is $s^2 + 4s - 5 = (s+5)(s-1)$. Thus, this singular point corresponds to two real eigenvalues with opposite sign, that is, a saddle point. The corresponding eigenvectors in the state-space are easily computed to be the lines passing through the point $(2,0)$ with slopes -5 and 1, respectively.

From this analysis, we are now able to predict the local behavior of the system around these two equilibria. However, the rest of the picture still remains unknown.

- **Plotting the isoclines**

  Remember that an isocline is the curve through which all trajectories go through with the same slope m. The equation for the isocline is therefore

$$m = \frac{\dot{x}_2}{\dot{x}_1} = \frac{-5x_1 - 4x_2 + 2.5x_1^2}{x_2}$$

  or

$$x_2 = \frac{2.5}{m+4}x_1(x_1 - 2).$$

  These are parabolas that all pass through the two singular points, and with an extremum at $x_1 = 1$. This is illustrated in Fig. 3.9. After plotting these trajectories, it now becomes clear which region of the plane leads towards the stable equilibrium and which leads to instability.

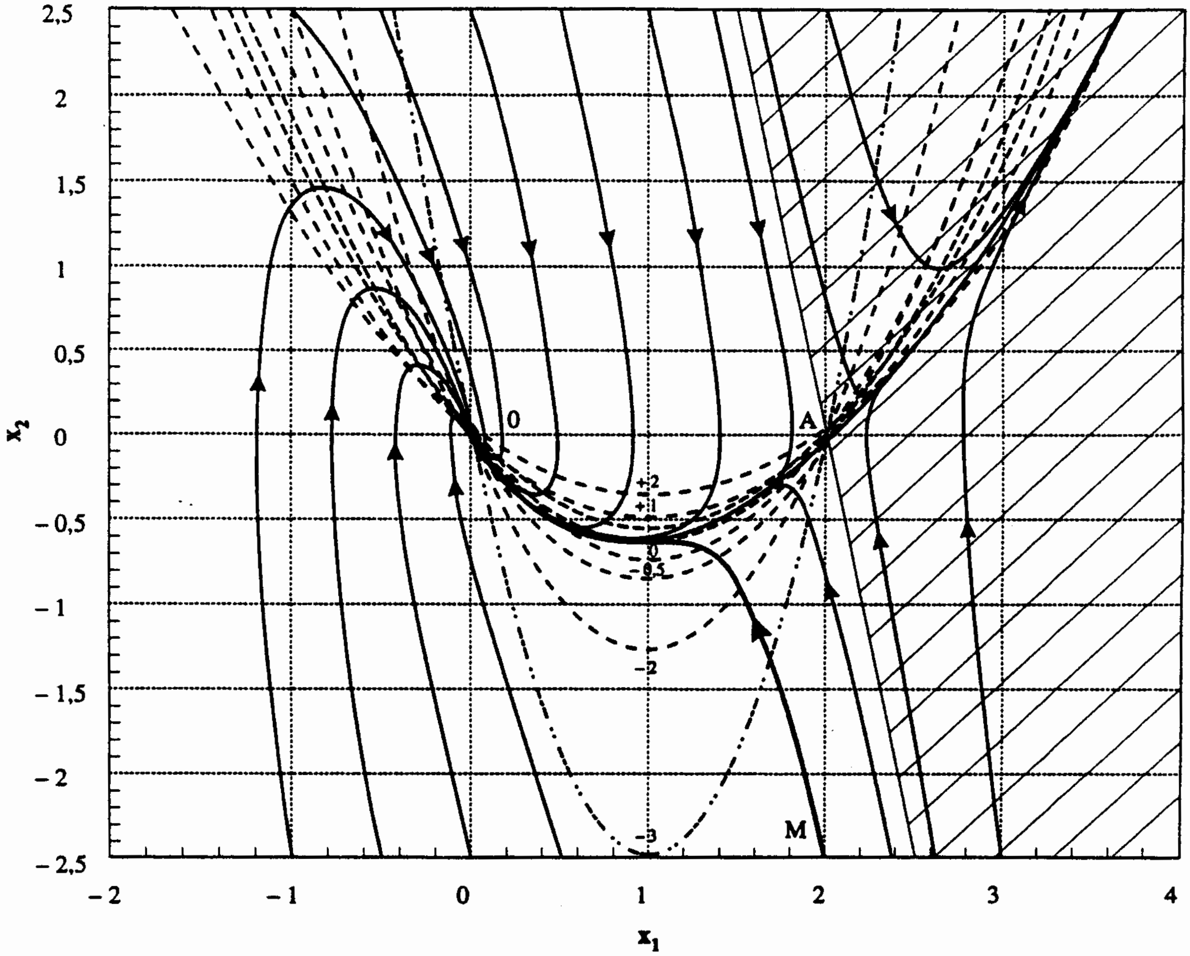

Figure 3.9: Complete phase-plane analysis

Problems

1. **Isoclines for linear, second-order systems**

   What is the shape of the isoclines for the second-order system

$$\begin{aligned}\frac{d}{dt}x_1 &= ax_1 + bx_2 \\ \frac{d}{dt}x_2 &= cx_1 + dx_2\end{aligned}$$

2. **Singular point**

   Consider the equation

$$\ddot{x} - 2\dot{x} - 1 = 0.$$

   Can you tell whether this system is stable? Draw typical trajectories, along with the principal directions for this system, in the canonical basis.

3. **Singular points for linear systems with pure eigenvalues**

   Plot the trajectories in the vicinity of the origin for the linear system

$$\ddot{y} + y = 0.$$

   What are these trajectories?

   Repeat this procedure for the system

$$\ddot{y} = 0,$$

   and

$$\ddot{y} + \dot{y} = 0,$$

   and

$$\ddot{y} - \dot{y} = 0.$$

   Have we now covered all possible cases?

4. **Stepper motor and why people like it**

   In this exercise, you are going to show why stepper motors are very nice, robust devices for control purposes.

(a) Consider the stepper motor shown in Fig. 3.6 and its free-motion differential equation

$$\ddot{\theta} + \lambda\dot{\theta} + \sin N\theta/2\pi = 0.$$

Draw the phase-plane portrait for this system, for $\lambda = 0.1, 0.5, 1$.

(b) We now assume that a voltage may be applied accross the rotor's armatures, such that the differential equation for its angular deviation becomes

$$\ddot{\theta} + \lambda\dot{\theta} + \sin N\theta/2\pi = u,$$

where u is the applied voltage. We want to control the stepper motor from one equilibrium to the next via impulses, that is, $u = u_0\delta(t)$ where $\delta(t)$ is the Dirac function. Assume the stepper motor is originally at rest. Give a range of values of u_0 such that the motor "falls" into the next equilibrium. Do this for $\lambda = 0.1, 0.5, 1$. Assume your voltage source is not really reliable. What value of u_0 and λ would you recommend?

(c) Assume now this motor was hooked on one side to a computer with a clock, on the other side to a set of demultiplying gears leading to a large telescope for star-following purposes. Would you trade your stepper motor for a beautiful, linearly behaving motor that works like

$$\ddot{\theta} + \lambda\dot{\theta} = u?$$

Justify your statement.

5. **A dynamical system with multiple trajectories**

Assume you are walking on a rope (modelled as a straight, horizontal line) for the first time. How would you best describe the dynamics for the altitude of your body as a function of time? (assume that once you fall off the rope, you can't cling on it anymore).

6. **Explosions**

Draw the phase-plane portrait of the dynamical system

$$\frac{d}{dt}x = x^2.$$

Also, time your trajectories using the techniques described in this chapter.