Variational formulation

Hu-Washizu principle:

$$\boxed{J[\varphi, F, P] = \int_{B_0} \left[W(F) + P_{iJ} (\varphi_{i,J} - F_{iJ}) - \rho_0 B_i \varphi_i \right] dV_0 - \int_{S_{01}} N_J P_{iJ} (\varphi_i - \bar{\varphi}_i) \, dS_0 - \int_{S_{02}} \bar{T}_i \varphi_i \, dS_0}$$

$$\text{BVP} \iff \langle DJ[\varphi, F, P], (\eta, A, B)\rangle = 0 \quad \forall \eta, A, B$$

Minimum potential energy principle

assume compatibility and constitutive hold.

$$\rightarrow J(\varphi) = \int_{B_0} \left[W(\nabla_0 \varphi) - \rho_0 B_i \varphi_i \right] dV_0 - \int_{S_{02}} \bar{T}_i \varphi_i \, dS_0$$

$$\text{Equilibrium} \iff \langle DJ[\varphi], \eta\rangle = 0 \quad \forall \text{ admissible } \eta$$

$\eta = 0$ on S_{01}

$$\langle DJ[\varphi], \eta\rangle = \frac{d}{d\epsilon} J(\varphi + \epsilon \eta)\Big|_{\epsilon=0}$$

$$\langle DJ[\varphi], \eta\rangle = \int_{B_0} \left[\langle DW(F), \eta\rangle - \rho_0 B \cdot \langle D\varphi, \eta\rangle \right]$$

$$- \int_{S_{02}} \bar{T} \langle D\varphi, \eta\rangle \, dS_0$$

$$\langle D\varphi, \eta \rangle = \frac{d}{d\epsilon}(\varphi + \epsilon\eta)\Big|_{\epsilon=0} = \eta$$

$$\langle DW, \eta \rangle = \frac{\partial W}{\partial F_{iJ}} \langle D F_{iJ}, \eta \rangle$$

$$= P_{iJ} \frac{d}{d\epsilon}\Big[(\varphi_i + \epsilon \eta_i)_{,J}\Big]\Big|_{\epsilon=0}$$

$$= P_{iJ} \frac{d}{d\epsilon}\Big[\varphi_{i,J} + \epsilon\, \eta_{i,J}\Big]_{\epsilon=0}$$

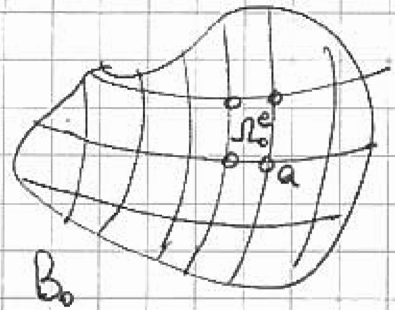
$$= P_{iJ}\, \eta_{i,J}$$

$$\boxed{\langle dJ[\varphi], \eta \rangle = \int_{B_0} \Big[P_{iJ}\, \eta_{i,J} - \rho_0 B_i \eta_i \Big] dV_0 - \int_{S_{02}} \bar{T}_i\, \eta_i\, dS_0}$$

$$= 0 \quad \forall \text{ admissible } \eta$$

$\Rightarrow \langle DJ[\varphi], \eta \rangle = 0 \equiv$ Principle of virtual work.

FINITE ELEMENT APPROXIMATIONS



$$B_0 = \bigcup_{e=1}^{E} \Omega_0^e \quad \text{(disjoint)}$$

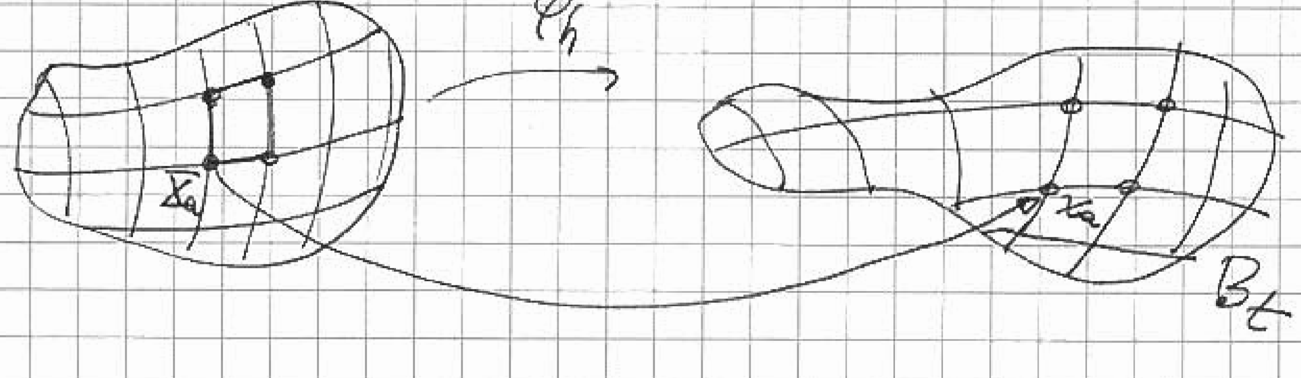
$a = 1, \dots, N \qquad \{\underline{X}_a\} \equiv$ nodal coordinates

Interpolate def mapping:

$$\boxed{\varphi_i(\underline{X}, t) \sim (\varphi_h)_i = \sum_{a=1}^{N} x_{ia}(t)\, N_a(\underline{X})}$$

undeformed or material shape functions

$\underline{x}_a(t) = \varphi_h(\underline{X}_a, t) \equiv$ def. nodal coordinates



Rayleigh-Ritz method

Formulate a discretized functional

$J_h(\varphi_h) = J(\varphi_h)$ make it stationary

$$\langle DJ(\varphi_h), \eta_h \rangle = 0 \quad \forall \text{ admissible } \eta_h$$

→ finite dimensional problem → $\boxed{\dfrac{\partial J_h}{\partial x_{ia}} = 0}$

$$\boxed{J_h[\varphi] = \int_{B_0} \left[W(F_h) - \rho_0 B_i \varphi_{hi} \right] dV_0 - \int_{S_{02}} \bar{T}_i \varphi_{hi} \, dS_0}$$

where $\boxed{F_h = \nabla_0 \varphi_h}$

$$(F_h)_{iJ} = \left(\sum_{a=1}^{N} x_{ia} N_a \right)_{,J} = \sum_{a=1}^{N} x_{ia} N_{a,J}$$

$$\int_{B_0} \left[W\left(\sum_{a=1}^{N} x_{ia} N_{a,J} \right) - \rho_0 B_i \left(\sum_{a=1}^{N} x_{ia} N_a \right) \right] dV_0$$

$$- \int_{S_{02}} \bar{T}_i \left(\sum_{a=1}^{N} x_{ia} N_a \right) dS_0$$

$$\frac{\partial W(F_h)}{\partial x_{ia}} = \frac{\partial W}{\partial F_{iJ}}(F_h)\, N_{a,J} = P_{iJ}(F_h)\, N_{a,J}$$

$$\frac{\partial J_h}{\partial x_{ia}} = \boxed{\underbrace{\int_{B_0} \left[P_{iJ}(F_h)\, N_{a,J} - \rho_0 B_i N_a \right] dV_0}_{F^{int}(\underline{x})} \underbrace{- \int_{S_{02}} \bar{T}_i N_a \, dS_0}_{-F^{ext}(\cancel{x})} = 0}$$

$$\frac{\partial J_h}{\partial x_{ia}} = 0 \iff \boxed{F^{int}(\underline{x}) = F^{ext}(\cancel{x})}$$

Linear elasticity: $\left.\begin{aligned} F^{int} &= Ku \\ F^{ext} &= f \end{aligned}\right\} Ku = f$

Galerkin approach   PVW + discretization + symmetric weighting

$$\left(\eta_h = \sum_{a=1}^{N} \eta_{ia} N_a \right)$$

get the same equations.

Diagramm commutes:

Rayleigh Ritz

$$\begin{tikzcd}
J \arrow[r, "\text{discretize}"] \arrow[d, "\text{``linearize''}"'] & J_h \arrow[d, "\text{``linearize''}"] \\
\langle DJ, \eta \rangle = 0 \arrow[r, "\text{discretize}"'] & \langle DJ_h, \eta_h \rangle = 0
\end{tikzcd}$$

Galerkin

$$\underbrace{\sum_{e=1}^{E} \int_{\Omega_0^e} P_{iJ}(F_h)\, N^e_{a,J}\, dV_0}_{F^{int}(\underline{x})} \underbrace{- \sum_{e=1}^{E} \int_{\Omega_0^e} \rho_0 B_i N_a^e\, dV_0 - \sum_{e=1}^{E} \int_{\partial\Omega_0^e \cap S_{02}} \bar{T}_i N_a^e\, dS_0}_{F^{ext}(t)} = 0$$