Linear elasticity:

Dirichlet form: $a(u,v) = \int_B C_{ijkl}\, u_{i,j}\, u_{k,l}\, dV$

Energy norm: $\|u\|_E = \left[a(u,u)\right]^{1/2}$

If $C_{ijkl}(x) \in L^\infty(B)$, coercive, convex $\Rightarrow$

$$c\|u\|_1 \leq \|u\|_E \leq C_1 \|u\|_1$$

Again: $\|u-u_h\| \leq \inf_{v_h\in V_h} \|u-v_h\|_E$ (best approx.) (of FE solution)

$$\Rightarrow \|u-u_h\|_E \leq \|u-u_I\|_E \leq C\, \|u-u_I\|_1$$

$$\Rightarrow \boxed{\|u-u_h\|_E \leq \sum_{e=1}^{E} C\, \sigma_h^e\, (h^e)^k\, |u^e|_{k+1}}$$

$(m=1)$
(order of derivative in Dirichlet form)

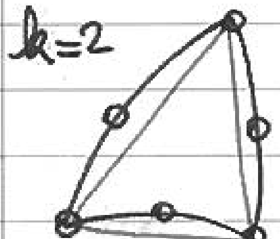
The previous estimate provides the rate of convergence as the mesh is refined ($h \to 0$). However a knowledge of the bound requires a knowledge of the unknown exact solution "u" (a priori error estimates). Want a posteriori

Need $|u^e|_{k+1}$, $e=1,\dots,E$

Cannot use $|u_h^e|_{k+1}$ since $=0$

$$(u_h^e \in P_k(\Omega_h^e))$$

- Assume $k \geqslant 2$



$u_h \in P_k(\Omega_h^e)$

$v_h \in P_{k-1}(\Omega_h^e)$: P_{k-1} interpolant to u_h

v_h also defines a converging approximation

$$\|u-v_h\|_E \leqslant \|u-u_h+u_h-v_h\|_E \leqslant \|u-u_h\|_E + \|u_h-v_h\|_E$$

$$\leqslant \sum_{e=1}^{E} C\sigma_h^e (h^e)^k |u^e|_{k+1} + \sum_{e=1}^{E} C\sigma_h^e (h^e)^{k-1} |u_h^e|_k$$

$\Rightarrow$ • $\|u - v_h\|_E \to 0$ as $h^e \to 0$

the first term converges faster, i.e.;

as $h^e \to 0$, $(h^e)^k$ cancels compared to $(h^e)^{k-1}$

$$\rightarrow \boxed{\|u - v_h\|_E \leq \sum_{e=1}^{E} C\, \sigma_h^e\, (h^e)^{k-1}\, \underline{\underline{|u_h^e|_k}}}$$

"A posteriori" error estimate, "local" can be computed element by element.

Numerical integration errors

Fully integrated case: $a(u_h, v_h) = \langle f, v_h \rangle \; \forall v_h \in V_h$

where $a(u,v) = \int_B C_{ijkl}\, u_{i,j}\, v_{k,l}\, dV$

When we introduce numerical quadrature we obtain:

$$\tilde{a}(u,v) = \sum_{e=1}^{E} \sum_{q=1}^{Q} W_q \left(C_{ijkl}\, u_{k,l}\, v_{i,j} \right)(\xi_q^e)$$

$$\tilde{a}(\tilde{u}_h, v_h) = \langle f, v_h \rangle \quad \forall v_h \in V_h$$

In general $\quad \tilde{u}_h \neq u_h$

We know that:

$$\|u - u_h\|_E \sim O(h^{k+1-m}) \text{ as } h \to 0$$

Under what conditions:

$$\|u - \tilde{u}_h\|_E \to 0 \text{ as } h \to 0 ?$$

At what rate?

Strang & Fix, p. 181

Proposition: • Assume $\exists\, c > 0 \,/\, \tilde{a}(u_h, u_h) \geq c \|u_h\|_m \quad \forall u_h \in V_h$

(definiteness)

• Assume: (exact quadrature for special cases)

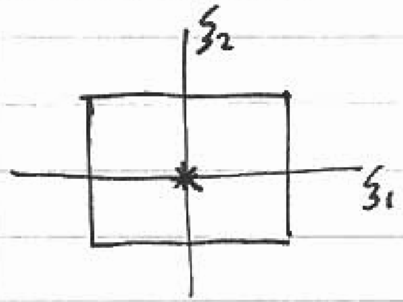
$$\sum_{q=1}^{Q} W_q \left[C_{ijkl}\, \varepsilon_{kl}\, v_{i,j} \right](\xi_q) = \int_\Omega C_{ijkl}\, \varepsilon_{kl}\, v_{i,j}\, d\Omega$$

$\forall \varepsilon_{kl} \in P_{l-m}(\Omega), \ \forall v_h \in V_h$

$$\Rightarrow \boxed{\|u - \tilde{u}_h\|_m \sim O(h^{l+1-m}) \text{ as } h \to 0}$$

- **Corollary:** $\ell = k$ for same rate of convergence as fully integrated solution

Examples:

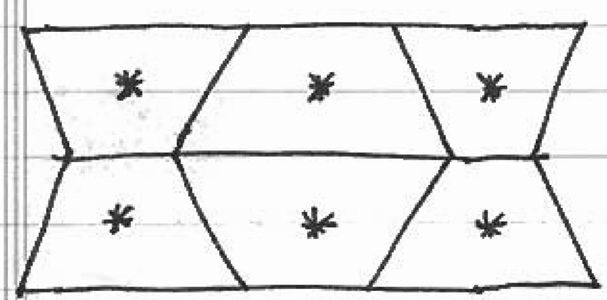


Numerical quadrature must be exact for:

$$\varepsilon_{ij} \in P_0(\Omega)$$

(constant strains)

It seems it would be enough to use "one" quadrature point. However:



(hourglass mode)

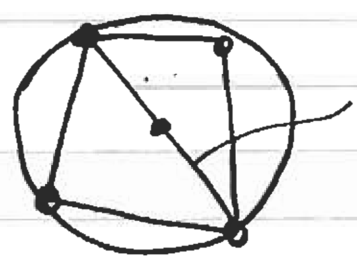
$$\boxed{\begin{array}{l} \varepsilon_{ij}(\xi_q) = 0 \Rightarrow \\ \tilde{a}(u_h, u_h) = 0 \end{array}}$$

"spurious zero energy mode"

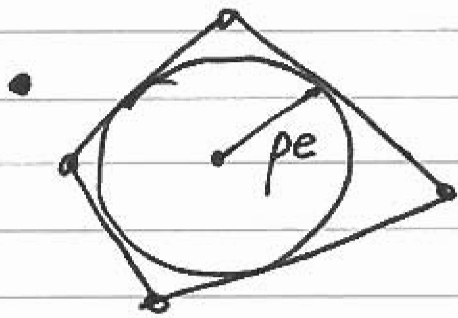
$\tilde{a}$ is not positive definite

Check always that "K^e" has no zero energy modes other than rigid body modes.

Notation:

- "m" order of derivatives in Dirichlet form. ($m=1$ for linear elasticity)
  ($m=2$ for plate theory)
- "k" highest order of polynomials fully included in interpolation
- 

  h^e: radius of smallest circumscribed circle
  ↳ element size



ρ^e = biggest circle inscribed in the element.

Basic error estimates

$$\|u - u_h\| \leq C(u) \frac{h^{k+1}}{\rho^m}$$

$h = h^e$, $\rho = \rho^e$ } for "e" such that $\frac{(h^e)^{k+1}}{(\rho^e)^m}$ is maximum

Define $\sigma^e = \frac{h^e}{\rho^e}$ element aspect ratio

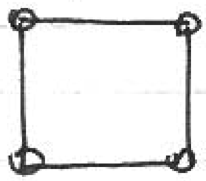
$$\|u - u_h\| \leq C(u) \frac{h^{k+1}}{(h^e)^m} (\sigma^e)^m \leq C(u)\, h^{k+1-m} (\sigma)^m$$

Where $\sigma_h \leq \sigma$, assume regular refinements

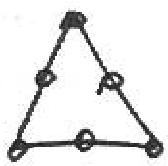
$$\|u - u_h\| \leq C'(u)\, h^{k+1-m}$$

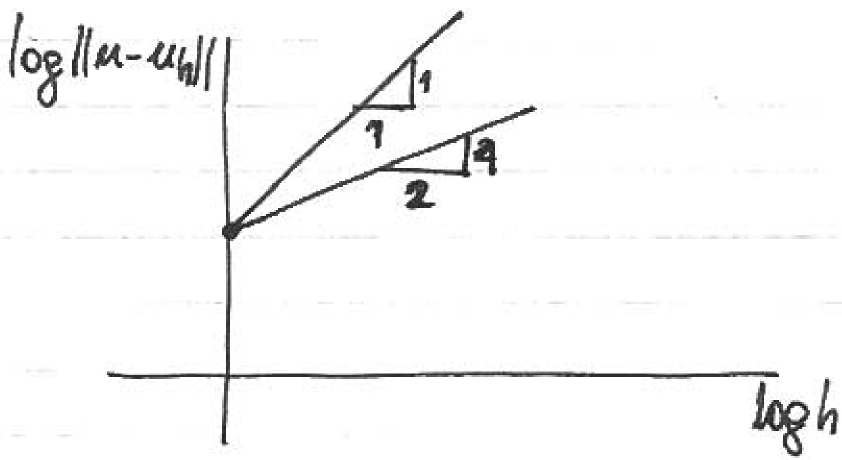
$\longrightarrow$ $\boxed{k+1-m \equiv \text{rate of convergence}}$

Examples

-  linear elasticity $k=1$, $m=1$

$$\|u-u_h\| = C'(u)\,h$$

-  $k=2$, $m=1$ $\quad \|u-u_h\| = C'(u)\,h^2$



Conditions for convergence

(1) $\|u_h\| < \infty$, otherwise $\|u-u_h\| \to \infty$

Finite element interpolation must give u_h with finite energy.

Sufficient conditions $\quad N_a \in C^m(\Omega^e)$

$$N_a \in C^{m-1}(\partial\Omega^e_{int})$$

Linear elasticity: $N_a \in C^1(\Omega^e)$
$N_a \in C^0(\partial\Omega^e)$ interior

Beam theory: $N_a \in C^2(\Omega^e)$
$N_a \in C^1(\partial\Omega^e)$ interior

~~Elasticity: $m=1 \Rightarrow k>0$, "k" at least~~ 1

~~Patch Test~~

② $\|u-u_h\| \leq C'(u)\, h^{k+1-m}$

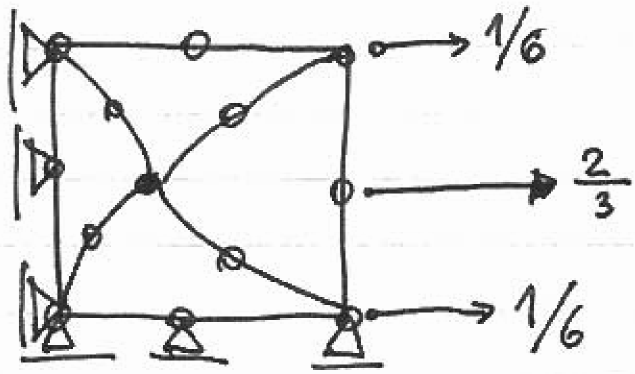
For convergence $\boxed{k+1-m>0}$

for fixed "m" $\boxed{k>m-1}$ COMPLETENESS

Elasticity: $m=1 \Rightarrow k>0$, "k" at least 1.

Patch test: Completeness $\Rightarrow$

constant strain state must be included exactly in the interpolation (up to machine precision)



$\Rightarrow \varepsilon_{11} = \text{const.}$

check for exact $u_i, \varepsilon_{ij}, \sigma_{ij}$

- $\|u - u_h\| \leq C(u)\, \sigma^m\, h^{k+1-m}$

$$C(u) = C \; |u|_{m+1}$$

$$|u|_{m+1} = \left[\int_B |D^{m+1} u|^2 \, dV\right]^{1/2}$$

strain gradients

$\Rightarrow$ high strain-gradients in exact solution slow down convergence.