173B

Multidimensional case

$A\dot{y} + By = 0$ (unforced) , $\begin{bmatrix} A = A^T \\ A > 0 \\ \text{sym } B > 0 \end{bmatrix}$

$y(0) = y_0$

Algorithm: $F(\Delta t)$, consistent

$$y_n = F^n(\Delta t)\, y_0$$

Introduce energy norm: $\|y\| = \sqrt{y^T A y}$

- Now the case $A \neq I$:

$$A\dot{y} + By = 0 \implies \tilde{y} = A^{1/2} y$$

$$\dot{\tilde{y}} + \tilde{B}\tilde{y} = 0 \qquad \tilde{B} = A^{-1/2} B A^{-1/2}$$

$$\|\tilde{B}\| = \sup_{\tilde{y}} \frac{\sqrt{\tilde{y}^T \tilde{B}^T \tilde{B} \tilde{y}}}{\sqrt{\tilde{y}^T \tilde{y}}} \quad \text{undo change}$$

$$\tilde{y}^T \tilde{B}^T \tilde{B} \tilde{y} = y^T \underbrace{A^{+T/2} A^{-T/2}}_{I} B^T \underbrace{A^{-T/2} A^{-1/2}}_{A^{-1}} B \underbrace{A^{-1/2} A^{1/2}}_{I} y$$

$$= y^T B^T A^{-1} B y$$

$$\tilde{y}^T \tilde{y} = y^T A^{T/2} A^{1/2} y = y^T A y$$

$$\|\tilde{B}\| = \sup_{A^{1/2}y} \underbrace{\frac{\sqrt{y^T B^T A^{-1} B y}}{\sqrt{y^T A y}}}$$

Rayleigh quotient corresponding to EVP:

$$\boxed{\Rightarrow \|\tilde{B}\| = \max_r |\lambda_r| \quad \text{where } \lambda_r\text{'s come from } (B^T A^{-1} B - \lambda^2 A) q = 0}$$

$$\frac{d}{dt}\|y\|^2 < 0 \quad , \quad \text{dissipative system}$$

$$\|y_n\| = \|F^n(\Delta t)\, y_0\| < \|y_0\|$$

Need norms for matrices:

Start with • $\dot{y} + By = 0$ $\quad (A = I)$

$$\Longrightarrow \|y\| = \sqrt{y^T y}$$

Definition:

$$\boxed{\|B\| = \sup_y \frac{\|By\|}{\|y\|}}$$

$$\Longrightarrow A = I \qquad \|B\| = \sup_y \underbrace{\frac{\sqrt{y^T B^T B y}}{\sqrt{y^T y}}}$$

Rayleigh quotient for eigenvalue problem:

$$\left(B^T B - \lambda^2 I\right)\varphi = 0$$

$$\Longrightarrow \quad \boxed{\|B\| = \max_r |\lambda_r|}$$

Spectral radius: $\rho(M) = \max\{\lambda / \det(M - \lambda A) =$

(not absolute value) $\Rightarrow \rho(M) \leq \|M\|$

e.g.: $B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \Rightarrow \begin{array}{l} \rho(B) = 0 \\ \|B\| = 1 \end{array}$

For symmetric positive definite matrices

$$B = B^T, \; B > 0 \Rightarrow \rho(B) = \|B\|$$

It can be shown that

$$\|B\| = \sup_y \frac{\|By\|}{\|y\|} = \max_r |\lambda_r|$$

defines a norm in the space of square matrices, i.e.:

① $\|B\| \geq 0$, $\|B\| = 0$ iff $B = 0$

② $\|aB\| = |a| \, \|B\|$

③ $\|B_1 + B_2\| \leq \|B_1\| + \|B_2\|$

④ $\|B_1 B_2\| \leq \|B_1\| \, \|B_2\|$

From these:

$$\begin{cases} \bullet\ \|By\| \leqslant \|B\|\,\|y\| \\ \bullet\ \|B^n\| \leqslant \underbrace{\|B\|\,\|B\| \ldots \|B\|}_{n \text{ times}} = \|B\|^n \end{cases}$$

Returning to stability condition

$$\|y_n\| \leqslant \|y_0\|$$

$$\|F^n(\Delta t)\, y_0\| \leqslant \|F^n(\Delta t)\|\, \|y_0\| \leqslant \|F(\Delta t)\|^n\, \|y_0\|$$

$\Rightarrow$ <u>sufficient condition</u> for $\|y_n\| \leqslant \|y_0\|$ is that

$$\boxed{\|F(\Delta t)\| \leqslant 1} \qquad \text{STABILITY CONDITION}$$

Another interpretation in terms of <u>perturbation of initial conditions</u>

$$A\dot{y} + By = 0 \qquad \left.\begin{array}{l} y(0) = y_0^{(1)} \\ y(0) = y_0^{(2)} \end{array}\right] \text{two different Initial conditions}$$

$$y_n^{(1)} = F^n(\Delta t)\, y_0^{(1)} \quad , \quad y_n^{(2)} = F^n(\Delta t)\, y_0^{(2)}$$

$$y_n^{(2)} - y_n^{(1)} = F^n(\Delta t)\left(y_0^{(2)} - y_0^{(1)}\right)$$

$$\begin{aligned}
\| y_n^{(2)} - y_n^{(1)} \| &\leqslant \| F^n(\Delta t)\left(y_0^{(2)} - y_0^{(1)}\right)\| \\
&\leqslant \| F^n(\Delta t)\| \, \| y_0^{(2)} - y_0^{(1)} \| \\
&\leqslant \| F(\Delta t)\|^n \, \| y_0^{(2)} - y_0^{(1)} \| \\
&\leqslant \| y_0^{(2)} - y_0^{(1)} \| \quad \text{if } \underline{\| F(\Delta t)\| < 1}
\end{aligned}$$

Definition: <u>Contractive mapping</u>

$$\boxed{\| y_{n+1}^{(2)} - y_{n+1}^{(1)} \| \leqslant \| y_n^{(2)} - y_n^{(1)} \|}$$

atenuates initial noise

Therefore: • stability ⟹ numerical solution is a <u>continuous</u> function of the initoal conditions ⟹ algorithm is <u>well-posed.</u>

<u>Lax equivalence theorem</u>

CONSISTENCY + STABILITY ⟺ CONVERGENCE

Definition: Truncation error

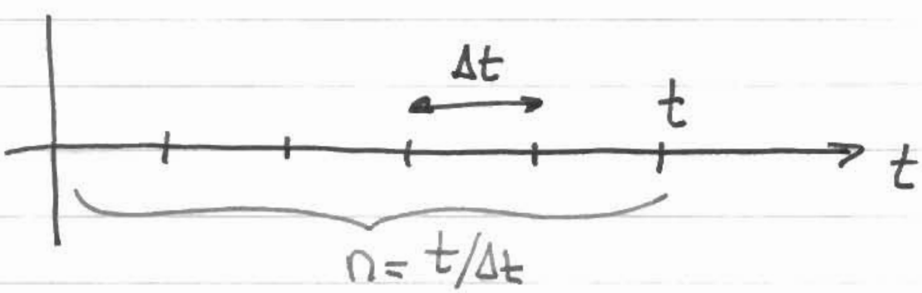
$y_{n+1} = F(\Delta t)\, y(t_n)$ — propagates exact solution at t_n numerically to t_{n+1} (approximate)

$y(t_{n+1}) = \underbrace{F(\Delta t)\, y(t_n)}_{y_{n+1}} + \underbrace{\tau(\Delta t)}_{\text{TRUNCATION ERROR}}$ — exact solution at t_{n+1} expressed as approximate plus truncation error

$$\boxed{\tau(\Delta t) = y(t_{n+1}) - y_{n+1}}$$

Consistency: $\|\tau(\Delta t)\| \sim O(\Delta t^2)$

Let "t" be a fixed time



(H) $\begin{cases} ① \ \|F(\Delta t)\| < 1 \quad \text{(stability)} \\ ② \ \|\tau(\Delta t)\| \leq C\, \Delta t^{(k+1)},\ k>0;\ \underline{k \equiv \text{order of accuracy}} \\ ③ \ e_0 = 0 \quad \text{error in the initial conditions} \end{cases}$

(T) $\Longrightarrow$ $\boxed{e(t) = y_{t/\Delta t} - y(t) \to 0 \quad \text{as } \Delta t \to 0}$

(P) $y_{n+1} = F(\Delta t)\, y_n$ error has accumulated from $y_0 \to y_n$

$$y(t_{n+1}) = F(\Delta t)\, y(t_n) + \tau(\Delta t)$$

$$\underbrace{y(t_{n+1}) - y_{n+1}}_{-e_{n+1}} = F(\Delta t) \underbrace{\left(y(t_n) - y_n\right)}_{-e_n} + \tau_n(\Delta t)$$

i.e.: error has two components:

- truncation error incurred in this time step
- amplification of error accumulated in all previous time steps.

Recurse to express total error in terms of individual truncation errors incurred in each time step:

$$e_{n+1} = F(\Delta t)\, e_n - \tau_n(\Delta t)$$

$$= F(\Delta t)\left[F(\Delta t)\, e_{n-1} - \tau_{n-1}(\Delta t)\right] - \tau_n(\Delta t)$$

$$= F^2(\Delta t)\, e_{n-1} - F(\Delta t)\, \tau_{n-1}(\Delta t) - \tau_n(\Delta t)$$

$$= F^2(\Delta t)\left[F(\Delta t)\, e_{n-2} - \tau_{n-2}\right] - F(\Delta t)\,\tau_{n-1}(\Delta t) - \tau_n(\Delta t)$$

$$= F^3(\Delta t)\, e_{n-2} - F^2(\Delta t)\,\tau_{n-2} - F(\Delta t)\,\tau_{n-1}(\Delta t) - \tau_n(\Delta t)$$

This expression says: as contributions to the error "e_{n+1}", the truncation error "τ_n" is not amplified, the truncation error: "τ_{n-1}" is amplified once,
"τ_{n-2}" - " twice

Iterating "n-2" more times (a total of "n" times)

$$= F^{3+n-2}(\Delta t)\; e_{n-2-(n-2)} - F^n(\Delta t)\,\tau_0 - \cdots\cdots$$

$$- F^2(\Delta t)\,\tau_{n-2} - F(\Delta t)\,\tau_{n-1}(\Delta t) - \tau_n(\Delta t)$$

$$= F^{n+1}(\Delta t)\,\underbrace{\cancel{e_0}}_{0\ \text{(H3)}} - \sum_{i=0}^{n} F^i(\Delta t)\,\tau_{n-i}$$

$$\boxed{e_{n+1} = -\sum_{i=0}^{n} F^i(\Delta t)\,\tau_{n-i}}$$

Estimate size of "e_n"

$$\|e_n\| = \Big\|\sum_{i=0}^{n-1} F^i(\Delta t)\,\tau_{n-i}\Big\| \leq \quad \text{property of norm}$$

$$\sum_{i=0}^{n-1} \|F^i(\Delta t)\,\tau_{n-i}\| \leq \quad \text{property of norm}$$

$$\leq \sum_{i=0}^{n-1} \|F^i(\Delta t)\| \, \|\tau_{n-i}\| \leq \qquad \text{(property of norm)}$$

$$\sum_{i=0}^{n-1} \|F(\Delta t)\|^i \, \|\tau_{n-i}\| \leq \qquad (\|F(\Delta t)\| \leq 1, \text{ stability})$$

$$\sum_{i=0}^{n-1} \|\tau_{n-i}\| \leq$$

$$\sum_{i=0}^{n-1} C(\Delta t)^{k+1} = n\,C(\Delta t)^{k+1}$$

$$= \underbrace{n\Delta t}_{t}\, C\, \Delta t^{k}, \quad k > 0$$

$$\Longrightarrow \quad \boxed{\|e_n\| \longrightarrow 0 \quad \text{as } \Delta t \longrightarrow 0 \qquad \text{qed}}$$