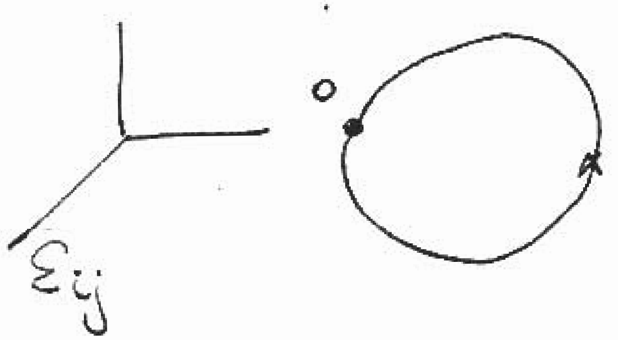
Elastic solids

Deformation power $= \sigma_{ij}\,\dot{\varepsilon}_{ij}$



cycle of deformation

$\Gamma_{(\tau)} : \varepsilon_{ij}(t)$

$t \in [0, T] \quad \varepsilon_{ij}(T) = \varepsilon_{ij}(0)$

$$\oint_{\Gamma} \sigma_{ij}\,\dot{\varepsilon}_{ij}\,dt = 0 \quad \forall \Gamma$$

$$\Leftrightarrow \exists\, W(\varepsilon_{ij}) \;/\; \sigma_{ij} = \frac{\partial W}{\partial \varepsilon_{ij}}$$

W: strain energy density

Conservation of energy:

$$\dot{u} = \sigma_{ij}\,\dot{\varepsilon}_{ij}$$

$$= \frac{\partial W}{\partial \varepsilon_{ij}}\,\dot{\varepsilon}_{ij} \quad \text{(elastic)}$$

$$\Longrightarrow \quad u = W(\varepsilon) + C$$

Legendre Transformation:

$$\sigma_{ij} = \frac{\partial W(\varepsilon_{ij})}{\partial \varepsilon_{ij}}$$

$$\chi = \sigma_{ij}\,\varepsilon_{ij} - W(\varepsilon_{ij})$$

$$d\chi = d\sigma_{ij}\,\varepsilon_{ij} + \cancel{\sigma_{ij}\,d\varepsilon_{ij}} - \cancel{\frac{\partial W}{\partial \varepsilon_{ij}}\,d\varepsilon_{ij}}$$

$$\rightarrow \varepsilon_{ij} = \frac{\partial \chi}{\partial \sigma_{ij}}$$

χ : Complementary strain energy density

~~Compatibility~~ Example: Thermoelasticity

$$W(\varepsilon, T) \qquad \sigma_{ij} = \frac{\partial W}{\partial \varepsilon_{ij}}$$

Linear thermoelasticity (Hooke's law):

$$W = \frac{1}{2}\left(\varepsilon_{ij} - \alpha_{ij} T\right) C_{ijkl} \left(\varepsilon_{kl} - \alpha_{kl} T\right)$$

W is quadratic $\Rightarrow \sigma_{ij} = C_{ijkl}\left(\varepsilon_{kl} - \alpha_{kl} T\right)$

$$C_{ijkl} = \frac{\partial^2 W}{\partial \varepsilon_{ij}\, \partial \varepsilon_{kl}} \qquad \text{elastic moduli}$$

symmetries: $\sigma_{ij} = \sigma_{ji} \Rightarrow C_{ijkl} = C_{jikl}$ 54c.

$\varepsilon_{ij} = \varepsilon_{ji} \Rightarrow C_{ijlk} = C_{ijkl}$ 36c.

$$\frac{\partial^2 W}{\partial \varepsilon_{ij}\, \partial \varepsilon_{kl}} = \frac{\partial^2 W}{\partial \varepsilon_{kl}\, \partial \varepsilon_{ij}} \Rightarrow C_{ijkl} = C_{klij} \quad 21c.$$

Isotropy:

Aris, R.: "Vectors, Tensors and the basic equations of fluid mechanics", Dover, 1989

Sokolnikoff: "Tensor Analysis: theory and applications to geometry and mechanics of continua" Wiley, 1964

$$C_{ijkl} = \lambda\, \delta_{ij}\, \delta_{kl} + \mu \left(\delta_{ik}\, \delta_{jl} + \delta_{il}\, \delta_{jk} \right)$$

λ, μ Lamè constants

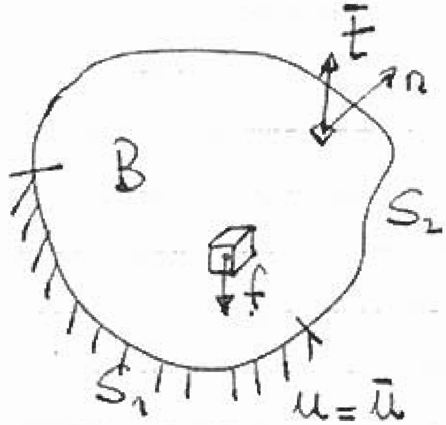
thermal isotropy: $\alpha_{ij} = \alpha\, \delta_{ij}$

$$\sigma_{ij} = \lambda\, \varepsilon_{kk}\, \delta_{ij} + \mu \left(\varepsilon_{ij} + \varepsilon_{ji} \right) - \alpha T \left(\lambda\, \delta_{ij} 3 + \mu 2 \delta_{ij} \right)$$

$$W = \frac{1}{2}\sigma_{ij}\varepsilon_{ij} = \frac{1}{2}\left[\lambda\,\varepsilon_{kk}\,\delta_{ij}\,\varepsilon_{ij} + 2\mu\,\varepsilon_{ij}\,\varepsilon_{ij}\right]$$
$$= \frac{1}{2}\lambda\,\varepsilon_{kk}^2 + \cancel{2}\mu\,\varepsilon_{ij}\,\varepsilon_{ij}$$

* Ex: Show $\chi(\sigma) = \frac{1}{2} C^{-1}_{ijkl}\,\sigma_{ij}\,\sigma_{kl} + \sigma_{ij}\,\alpha_{ij}\,T$ (Sdkdnikof,

Summary of field equations of linearized elasticity



$S = \partial B = S_1 \cup S_2$, $S_1 \cap S_2 = \emptyset$

S_1: displacement boundary

S_2: traction boundary

Equilibrium

$$\sigma_{ij,j} + f_i = 0 \quad \text{in } B$$
$$\sigma_{ij}\,n_j = \bar{t}_i \quad \text{on } S_2$$

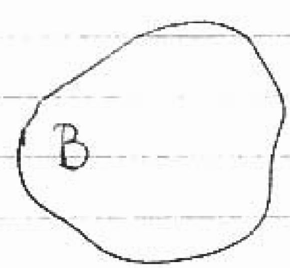
Compatibility

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) \quad \text{in } B$$
$$u = \bar{u} \quad \text{on } S_1$$

Constitutive relations $\quad \sigma_{ij} = \dfrac{\partial W}{\partial \varepsilon_{ij}} \quad , \quad \varepsilon_{ij} = \dfrac{\partial \mathcal{X}}{\partial \sigma_{ij}}$

Variational calculus

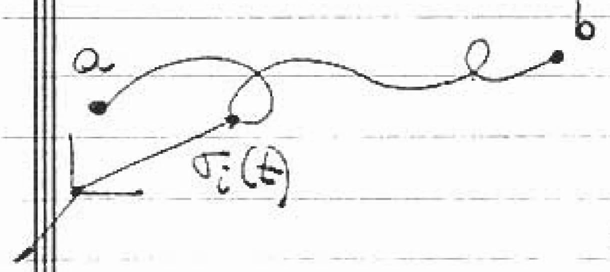
- I. M. Gelfand & S. V. Fomin, "Calculus of variations" Prentice Hall, 1963
- J. T. Oden, J. N. Reddy: "Variational Methods in Theoretical Mechanics", Springer-Verlag, 1983
- M. M. Vainberg: "Variational Methods in the theory of nonlinear operators", Holden Day, 1964



Let "$u(x)$" be a field over B expressing some state of the solid.

Let $J(u)$ be a functional of "u" (e.g. linear momentum, energy, etc).

Example of a functional: String



$\sigma_i(t) : [0, T] \rightarrow \mathbb{R}^3$

$B = [0, T] \quad , \quad u = \sigma$

$\sigma(0) = a \quad \sigma(T) = b$

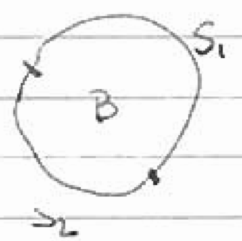
Length of string: $\underset{(\sigma)}{S} = \int_a^b ds = \int_0^T \frac{ds}{dt}\, dt = \int_0^T |\sigma'(t)|\, dt$

$J = S \,,\quad u = \sigma$

Extrema $\rightarrow$ calculus of variations

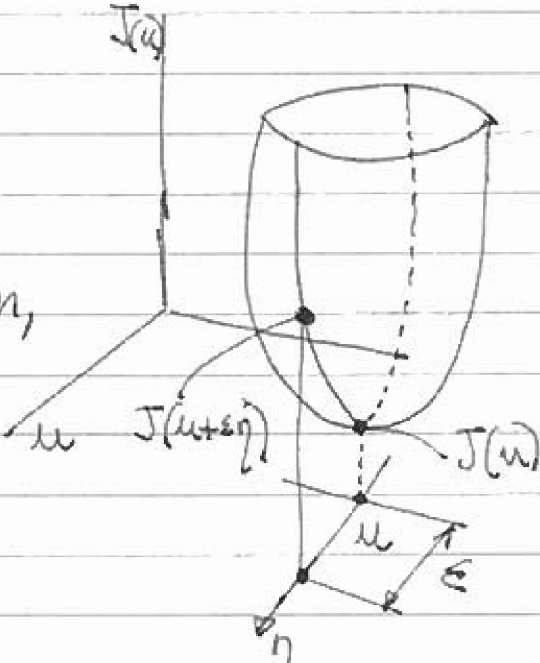
Given a functional $J(u)$, characterize those "u" which extremize J (for which J is either a maximum or a minimum.

Focus: $\boxed{J(u) = \int_B F(x, u, \nabla u)\, dV - \int_{S_2} \phi(x, u)\, ds}$



$S = S_1 \cup S_2 = \partial B$

$S_1 \cap S_2 = \phi$



Reduce to 1 variable problem, take derivatives $= 0$

Consider variations:

$u \rightarrow u + \varepsilon\eta$

$J(u + \varepsilon\eta) = J(\varepsilon)$

For u to be the minimizer of J: $\left.\frac{dJ(\varepsilon)}{d\varepsilon}\right|_{\varepsilon=0} = 0$

In more detail:

J stationary at u requires:

$$\left.\frac{dJ(u+\varepsilon\eta)}{d\varepsilon}\right|_{\varepsilon=0} = 0 \equiv \underbrace{\langle DJ(u), \eta \rangle}_{\text{first variation of } J \text{ in direction } \eta} = 0$$

$\forall$ admissible η

u satisfies essential boundary conditions on S_1

$$\left.\begin{aligned} u &= \bar{u} \text{ on } S_1 \\ u + \varepsilon\eta &= \bar{u} \text{ on } S_1 \end{aligned}\right\} \quad \boxed{\eta = 0 \text{ on } S_1}$$

admissible variations "η" must satisfy homogeneous boundary conditions over the essential boundary S_1.

$$\langle DJ(u), \eta\rangle = \frac{d}{d\varepsilon}\left\{\int_B F(x, u+\varepsilon\eta, \nabla(u+\varepsilon\eta))\, dV - \int_{S_2} \phi(x, u+\varepsilon\eta)\, dS\right\}\Bigg|_{\varepsilon=0}$$

$$= \left\{\int_B \left[\frac{\partial F}{\partial u_i}(x, u+\varepsilon\eta, \nabla(u+\varepsilon\eta))\, \frac{d}{d\varepsilon}(u_i+\varepsilon\eta_i) + \right.\right.$$

$$\left. + \frac{\partial F}{\partial u_{i,j}}(x, u+\varepsilon\eta, \nabla(u+\varepsilon\eta))\, \frac{d}{d\varepsilon}(u_{i,j}+\varepsilon\eta_{i,j})\right] dV -$$

$$\left. - \int_{S_2} \frac{\partial\phi}{\partial u_i}(x, u+\varepsilon\eta)\, \frac{d}{d\varepsilon}(u_i+\varepsilon\eta_i)\, dS\right\}\Bigg|_{\varepsilon=0}$$

$$\boxed{\langle DJ(u), \eta\rangle = \int_B \left[\frac{\partial F}{\partial u_i}\eta_i + \frac{\partial F}{\partial u_{i,j}}\eta_{i,j}\right] dV - \int_{S_2}\frac{\partial\phi}{\partial u_i}\eta_i\, dS}$$

J Stationary: $\langle DJ(u), \eta\rangle = 0 \quad \forall \eta$ admissible

Local form of stationarity condition

Integrate by parts term in $\eta_{i,j}$

$$\langle DJ(u), \eta\rangle = \int_B \left[\frac{\partial F}{\partial u_i}\eta_i - \left(\frac{\partial F}{\partial u_{i,j}}\right)_{,j}\right]\eta_i\, dV + \int_B \left(\frac{\partial F}{\partial u_{i,j}}\eta_i\right)_{,j} dV -$$

$$-\int_{S_2} \frac{\partial \phi}{\partial u_i} \eta_i \, ds$$

$$= \int_B \left[\frac{\partial F}{\partial u_i} - \left(\frac{\partial F}{\partial u_{i,j}} \right)_{,j} \right] \eta_i \, dv + \int_S \frac{\partial F}{\partial u_{i,j}} \eta_i \, n_j \, ds - \int_{S_2} \frac{\partial \phi}{\partial u_i} \eta_i \, ds$$

↳ can replace with S_2 since $\eta_i = 0$ on S_1

$$\Rightarrow \boxed{\left.\begin{aligned} & \frac{\partial F}{\partial u_i} - \left(\frac{\partial F}{\partial u_{i,j}} \right)_{,j} = 0 \quad \text{in } B \\ & \frac{\partial F}{\partial u_{i,j}} n_j = \frac{\partial \phi}{\partial u_i} \quad \text{on } S_2 \end{aligned}\right\} \text{Euler-Lagrange equations of } J(u)}$$