16.225 - Computational Mechanics of Materials
Homework assignment # 1
Handed out: 9/10/03
Due: 9/24/03

September 22, 2003

1. Verify that the Euler-Lagrange equations corresponding to the Hu-Washizu functional are the field equations of linear elasticity.

2. **Consistency test for constitutive models with a potential structure** (Computing assignment) Constitutive models that have a potential structure suggest a useful verification test of their implementations. The idea is to make use of the fact that in these models stresses and tangent moduli are obtained by direct differentiation of the strain energy density and compute the derived quantities in two different ways in order to compare them. The first way is directly from their implementation; the second, by numerical differentiation of the function. This is explained in more detail in the algorithm described in the following box:

(a) Generate random strain $\boldsymbol{\epsilon}$

(b) Compute the stresses $\sigma_{ij}(\boldsymbol{\epsilon})$ and the tangent moduli $C_{ijkl}(\boldsymbol{\epsilon})$ directly from their implementation

(c) Recompute these quantities by numerical differentiations, e.g., using central differences:

$$f'(a) = \frac{f(a+h) - f(a-h)}{2h} + \frac{h^2}{6} f'''(\eta)$$

for some $\eta \in [a-h, a+h]$. For the case of the stresses and moduli:

$$(\sigma_h)_{ij}(\boldsymbol{\epsilon}) \sim \frac{W(\boldsymbol{\epsilon} + h\mathbf{e}_i \otimes \mathbf{e}_j) - W(\boldsymbol{\epsilon} - h\mathbf{e}_i \otimes \mathbf{e}_j)}{2h}$$

$$(C_h)_{ijkl}(\boldsymbol{\epsilon}) \sim \frac{\sigma_{ij}(\boldsymbol{\epsilon} + h\mathbf{e}_k \otimes \mathbf{e}_l) - \sigma_{ij}(\boldsymbol{\epsilon} - h\mathbf{e}_k \otimes \mathbf{e}_l)}{2h}$$

(d) Test if:

$$\max_{ij} |\sigma_{ij}(\epsilon) - (\sigma_h)_{ij}(\epsilon)| < TOLE|\sigma(\epsilon)|$$

$$\max_{ijkl} |C_{ijkl}(\epsilon) - (C_h)_{ijkl}(\epsilon)| < TOLE|\mathbf{C}(\epsilon)|$$

- Implement a consistency test for a general linear-elastic material model. In this case:

$$W = W(\epsilon) = \frac{1}{2} C_{ijkl}\epsilon_{ij}\epsilon_{kl}$$

$$\sigma_{ij} = \frac{\partial W}{\partial \epsilon_{ij}}$$

$$C_{ijkl} = \frac{\partial^2 W}{\partial \epsilon_{ij} \partial \epsilon_{kl}}$$

- Apply your program to the case of an isotropic elastic material with constants: $E = 210GPa, \nu = 0.33$ and the following initial strains:

– simple tension
– simple shear

In both cases verify the correctness of the values of the stresses and tangent moduli produced by your program by comparing them with the exact values.