# 3D Boundary Layers.

A) Crossflow Instability

B) Transition Mechanisms Mack, AGARD-R-709 June 1984.

Reading: White 342 – 344. , Saric & Reed Annual Review 1989 (Stability of 3-D BLs)

A) 3D O-S Equations:

Recall, for mean flow we can add small perturbation, linearize and derive OS equation.

· incomp, laminar, visc NS

$$\nabla \cdot \vec{u} = 0$$

$$\frac{D\vec{u}}{Dt} = -\nabla p + \frac{1}{Re}\nabla^2 \vec{u}$$

$$\vec{u} = \vec{U} + \hat{\vec{u}}$$

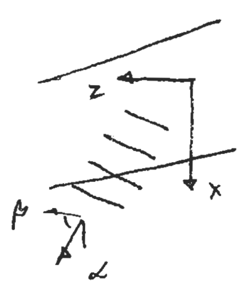
$$p = p_0 + \hat{p}$$

Subst and neglecting higher powers gives linearized PDE for disturbances.

Assume disturbance is of the form

$$\hat{\vec{u}} = \tilde{\vec{u}}(y)\, e^{i(\alpha x + \beta z - \omega t)} \quad , \quad \hat{p} = \tilde{p}(y)\, e^{i(\quad)}$$

General wave number vector $\alpha\hat{i} + \beta\hat{k}$



$$i\alpha\tilde{u} + \frac{\partial\tilde{v}}{\partial y} + i\beta\tilde{w} = 0$$

$$-i\omega\tilde{u} + i\alpha\tilde{u}U + i\beta\tilde{u}W + U'\tilde{v} = -\frac{i}{\rho}\alpha\tilde{p} + \frac{1}{Re}\left[D^2 - \alpha^2 - \beta^2\right]\tilde{u}$$

$$-i\omega\check{v} + i\alpha\check{v}U + i\beta\check{v}W = -\frac{1}{\rho}\frac{d}{dy}\check{p} + \frac{1}{Re}\left[D^2 - \alpha^2 - \beta^2\right]\check{v}$$

$$-i\omega\check{w} + i\alpha\check{w}U + i\beta\check{w}W + \frac{dW}{dy}\hat{v} = -\frac{i}{\rho}\beta\check{p} + \frac{1}{Re}\left[D^2 - \alpha^2 - \beta^2\right]\check{w}$$

$$-i\omega(\alpha\check{u} + \beta\check{w}) + i\alpha^2\check{u}U + i\alpha\beta\check{u}W + \alpha U'\check{v}$$

$$+ i\alpha\beta\check{w}U + i\beta^2\check{w}W + \beta W'\check{v}$$

$$\check{v}(\alpha U' + \beta W')$$

$$-i\omega\check{u} + i(\alpha^2\check{u}U + \beta^2\check{w}W) \ldots$$

$\vec{\alpha}' = \alpha\hat{i}$

$\alpha = \|\alpha\|\cos\phi$

Add $\alpha(1) + \beta(3) \Rightarrow$ we get.

$$\cos\phi = \frac{\alpha}{\sqrt{\alpha^2+\beta^2}}$$

$$i\alpha'\check{u}' + v' = 0$$

$$i\alpha'\check{u}'(U' - c) + U'\check{v}' = -\frac{i}{\rho}\alpha'\check{p} + \frac{1}{Re}\left[D^2 - \alpha'^2\right]\check{u}'$$

$\alpha = \alpha'\cos\phi$

$\beta = \alpha'\sin\phi$

$$i\alpha\check{v}(U' - c) = -\frac{1}{\rho}\frac{d\check{p}}{dy} + \frac{1}{Re}(D^2 - \alpha'^2)\check{v}$$

2D Eqns with $\check{u}'$, $U'$ // to wave vector direction

2D analysis for effective 2D flow along $\phi$

Inputs: $U(y)$, $W(y)$, $Re$ (eliminate pressure

Eigenvalue Prob: $\omega$ if $\alpha_r$, $\beta_r$ specified (temp)

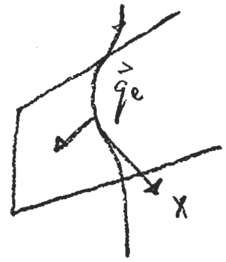
$\alpha$ if $\beta_r$, $\omega_r$ " (x-spatial)

$\beta$ if $\alpha_r$, $\omega_r$ specified (z-spatial)

2D result is $W = 0$.

Simplest to assume that $x$ - aligned with $\vec{q}_e$

$$U(y_e) = u_e \, , \quad W(y_e) = 0$$



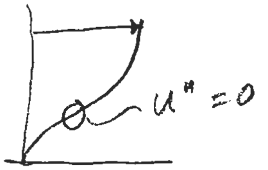

Fit in the framework of 3D approximate approaches discussed earlier

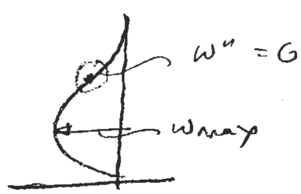
> Transition Mech : Streamwise instability : $\alpha_i < 0$ (T-S) waves
Crossflow " $\beta_i < 0$ (crossflow waves)

Calculation Results:

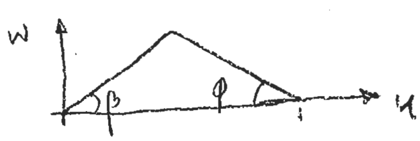
For $\frac{U_e \delta}{\nu} = R_{e\delta} > R_{\delta crit}$ , $\alpha_i < 0$ promoted by inflected $U(y)$ (adverse $\frac{dp}{dx}$ only)

For $\frac{W_{max}\delta}{\nu} > R_{crit}$ $\beta < 0$ promoted by inflected $W(y)$ (always !!)




Example : Johnston




$$W = \begin{cases} U \tan\beta \\ (1-U)\tan\phi \end{cases}$$

$$W'' = U'' \tan\beta \quad \text{and} \quad -U'' \tan\phi$$

Rescrit ≈ 520 for Blasius. 

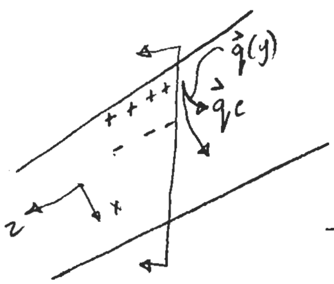
B> Transition Mechanisms

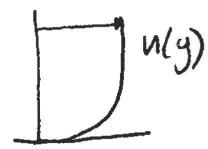
$\frac{W_{max}\delta}{\nu}$ strongly increased by sweep + favorable $\frac{dp}{dx}$

≈ 100 for instability with typical $W(y)$.

→ streamwise instability is suppressed in a strong favorable pressure gradient.

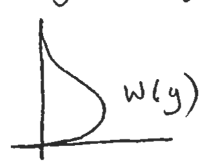
For a swept wing




→ fuller streamwise profile

→ strong crossflow

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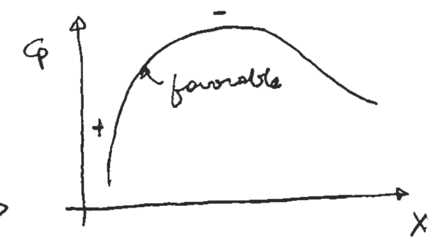
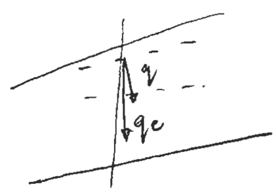


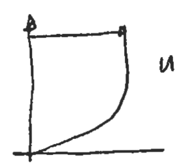

. streamwise inst. supp

. cross flow inst aggravated

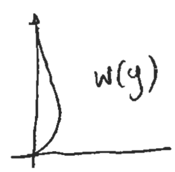
(Design tradeoff)

C>




— larger shape factor




weaker crossflow

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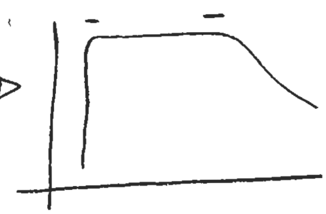
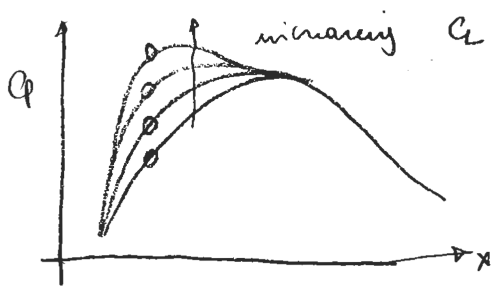


Drag polar of swept vs. unswept wing

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