# Stability and Transition

## 6.1> Small Perturbation Theory

A> Perturbation Flow Field

B> Linearization

C> Orr - Sommerfeld Eqn.

Reading: Sch 449 – 483
White 335 – 355

Steady laminar boundary layer flow when subject to small disturbances may become unstable (above a critical Reynolds) change/trans to turbulent flow. We would like to examine stability of the flow subject to small perturbations. Will they grow? unstable, or decay – stable.

Analogous to previous stability analyses, introduce small perturbation on mean flow.

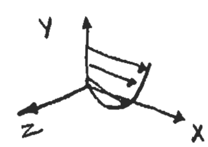
$$\nabla \cdot \vec{U}_0 = 0$$

$$\frac{D\vec{u}_0}{Dt} = -\nabla p_0 + \frac{1}{Re}\nabla^2 \vec{U}_0$$

Is mean flow stable to small disturbances.

$$\vec{u} = \vec{U}_0 + \vec{\hat{u}} \leftarrow (\hat{u}, \hat{v}, \hat{w})$$

$$p = p_0 + \hat{p}$$




where $|\vec{\hat{u}}| << |\vec{U}_0|$

Substitute above and neglect higher powers ~~of~~ $\vec{\hat{u}}$ & $\hat{p}$

$$\nabla \cdot \vec{\hat{u}} = 0$$

$$X \rightarrow \quad \frac{\partial \hat{u}}{\partial t} + U_0 \frac{\partial \hat{u}}{\partial x} + \hat{u}\frac{\partial U_0}{\partial x} + V_0 \frac{\partial \hat{u}}{\partial y} + \hat{v}\frac{\partial U_0}{\partial y} + W_0 \frac{\partial \hat{u}}{\partial z} + \hat{w}\frac{\partial U_0}{\partial z} = -\frac{\partial \hat{p}}{\partial x} + \frac{1}{Re}\nabla^2 \hat{u}$$

– linear PDE

Assume perturbed solution has the form

$$\vec{\hat{u}} = \vec{\breve{u}}(y)\, e^{i(\alpha x + \beta z - \omega t)} \qquad \hat{p} = \breve{p}(y)\, e^{i(\alpha x + \beta z - \omega t)}$$

Assume for simplicity that $\vec{U}_0 = (u_0(y), 0, 0)$ – 2D parallel flow

- approximate for Falkner-Skan flow $\frac{\partial u}{\partial x} \approx 0$
- exact for Poiseuille flow.

Linearized cont + mom simplifies to

$$\frac{\partial \hat{u}}{\partial x} + \frac{\partial \hat{v}}{\partial y} + \frac{\partial \hat{w}}{\partial z} = 0$$

$$\frac{\partial \hat{u}}{\partial t} + U_0 \frac{\partial \hat{u}}{\partial x} + \hat{v}\frac{\partial U_0}{\partial y} = -\frac{\partial \hat{p}}{\partial x} + \frac{1}{Re}\nabla^2 \hat{u}$$

$$\frac{\partial \hat{v}}{\partial t} + U_0 \frac{\partial \hat{v}}{\partial x} = -\frac{\partial \hat{p}}{\partial y} + \frac{1}{Re}\nabla^2 \hat{v}$$

$$\frac{\partial \hat{z}}{\partial t} + U_0 \frac{\partial \hat{w}}{\partial x} = -\frac{\partial \hat{p}}{\partial z} + \frac{1}{Re}\nabla^2 \hat{w}$$

Substitute $\vec{\breve{u}}(y)\, e^{i(\quad)}$

Note: $\frac{\partial}{\partial x}() = i\alpha(\ )$, $\frac{\partial}{\partial z} = i\beta(\ )$, $\frac{\partial}{\partial t} = i\omega()$

$$\Rightarrow \quad i\alpha\breve{u} + \frac{d}{dy}\breve{v} + i\beta\breve{w} = 0$$

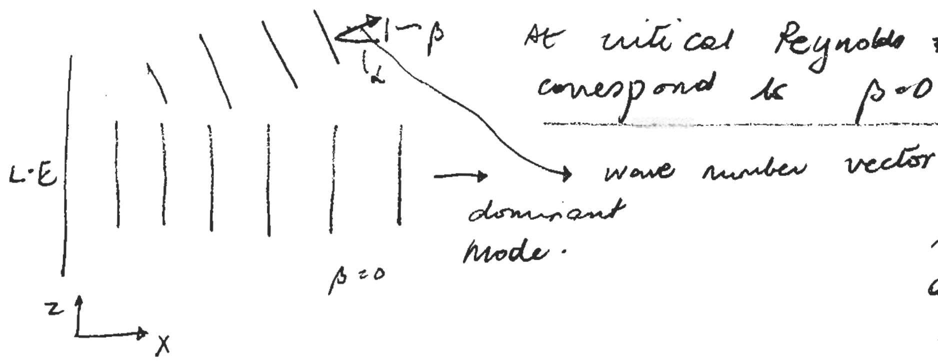
$$-i\omega\breve{u} + i\alpha U_0\breve{u} + \frac{dU_0}{dy}\breve{v} = -i\alpha\breve{p} + \frac{1}{Re}\left(\frac{d^2}{dy^2} - \alpha^2 - \beta^2\right)\breve{u}$$

$$-i\omega\breve{v} + i\alpha U_0\breve{v} = -\frac{d}{dy}\breve{p} + \frac{1}{Re}\left(\frac{d^2}{dy^2} - \alpha^2 - \beta^2\right)\breve{v}$$

$$-i\omega\hat{w} + i\alpha U_0\hat{w} = -i\beta\breve{p} + \frac{1}{Re}\left(\qquad\qquad\right)\breve{w}$$

Let $\beta = 0 \Rightarrow \hat{w} = 0$ (disturbance propagates in flow direction) ⑦

$\rightarrow$ Squire's theorem — worst case — lowest $Re_{crit}$.




At critical Reynolds #, unstable waves correspond to $\beta = 0$

$\rightarrow$ 2D disturbance is adequate to understand and model for practical engineering cases.

From continuity

$$\hat{u} = \frac{i}{\alpha}\frac{d}{dy}\hat{v}$$

Squire's transf:

$$\alpha' = \frac{\alpha}{\sqrt{\alpha^2+\beta^2}} \qquad R' = \frac{\alpha R}{\sqrt{\alpha^2+\beta^2}}$$

remove dependance on $\beta$.

$\therefore$ x-mom $\rightarrow$

$$i(\alpha U_0 - \omega)\frac{i}{\alpha}\frac{d}{dy}\hat{v} + \frac{d}{dy}U_0 \cdot \hat{v} = -i\alpha\hat{p} + \frac{1}{Re}(\quad)\frac{i}{\alpha}\frac{d}{dy}\hat{v}$$

$$\hat{p} = \frac{i}{i\alpha}\left[\frac{d}{dy}U_0\hat{v} - \frac{1}{\alpha}(\alpha U_0 - \omega)\frac{d}{dy}\hat{v} - \frac{1}{Re}\left(\frac{\partial^2}{\partial y^2} - \alpha^2\right)\frac{i}{\alpha}\frac{d}{dy}\hat{v}\right]$$

y-mom $\rightarrow$

$$+(\alpha U_0 - \omega)\hat{v} = -\frac{d}{dy}\hat{p} + \frac{1}{Re}\left(\frac{d^2}{dy^2} - \alpha^2\right)\hat{v}$$

Simplify,

$$\Rightarrow (\alpha U_0 - \omega)\left(\frac{d^2\hat{v}}{dy^2} - \alpha^2\hat{v}\right) - \alpha\frac{d^2U_0}{dy^2}\hat{v} + \frac{i}{Re}\left(\frac{d^4}{dy^4} - 2\alpha^2\frac{d^2}{dy^2} + \alpha^4\right)\hat{v} = 0$$

- Orr Sommerfeld eqn — 4th order complex ODE for $\hat{v}(y)$
- $U_0(y)$ is input (mean flow)
- $\omega = \alpha c$ — $\alpha$ wave number, $c$ wave speed.

~~Boundary Conditions~~

Boundary Conditions

Duct: $y=0 \quad \hat{v} = \hat{v}' = 0$

$y = 1 \ (h) \quad \hat{v} = \hat{v}' = 0$ (Poiseuille flow)

B-L: $y=0 \quad \hat{v} = \hat{v}' = 0$

$y = +\infty \quad \hat{v} = \hat{v}' = 0$

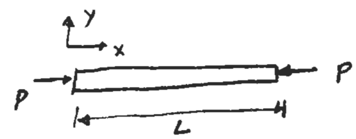
Free Shear layer: $y = \pm\infty$ " $= 0$.

Note: Governing ODE and boundary conditions are homogeneous

=> Eigenvalue problem: nontrivial $\hat{v}(y)$ exist only for certain combinations of $\omega$, $\alpha$, Re for a given mean flow $U_0(y)$

ample: Analogous to beam buckling (instability)

$$y'' + \frac{P}{EI} y = 0 \qquad y(0) = y(L) = 0$$




Solution: $y = A \sin(kx) + \cancel{B}^{\,0} \cos(kx)$ ← eigenfunctions

where $k^2 = P/EI$ $\quad y \neq 0$ only if $k = \pm 1, \pm 2, \dots$

↑ eigenvalues $\qquad$ A is arbitrary.

$k = \frac{n\pi}{L} \Rightarrow \frac{n^2\pi^2}{L^2} = P/EI$

$P_n = \frac{n^2\pi^2 EI}{L^2}$

$P_1 = \frac{\pi^2 EI}{L^2}$ (Euler buckling load)

Can only predict if unstable or not

In general, either $\alpha$ or $\omega$ will be complex
Two types of eigenvalue problems:

a) Temporal Amplification
b) Spatial Amplification

a) Temporal Amplification

$\alpha$ is real and specified
$\omega = \omega_r + i\omega_i$ will be calculated

$$e^{-i\omega t} = e^{-i\omega_r t} \cdot e^{\omega_i t}$$

↑ growth rate

$\omega_i > 0$ — growth in time
$\omega_i < 0$ — decay " "

$\hat{V}(y)$ - eigenfunctions (modes)

Solve $\mathcal{L}(\hat{V}(y), \omega; U(y), \alpha, Re) = 0$

Temporal analysis predicts behavior as $t \to \infty$ given initial disturbance / perturbation of $\alpha$. $\alpha_r$, $\alpha_i = 0$.

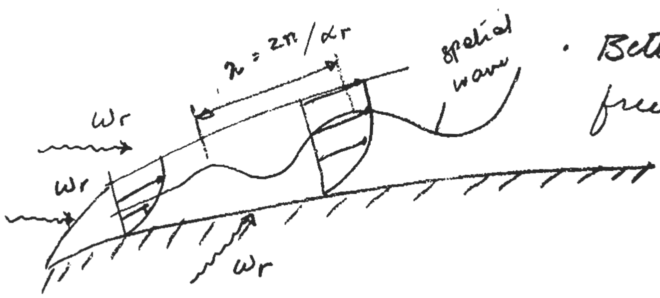
b) Spatial Amplification

$\omega$ real is specified and $\alpha = \alpha_r + i\alpha_i$ is to be calculated

$$e^{i\alpha x} = e^{i\alpha_r x} \cdot e^{-\alpha_i x}$$

↑ spatial growth rate

$\alpha_i < 0$ — growth downstream
$> 0$ — decay




· Better / more relevant to BL problem where free stream turbulence provides starting perturbation / disturbance

## Inviscid limit

i) $Re \to \infty$, we get Rayleigh Eqn.

$$(\alpha U_0 - \omega)(\hat{v}'' - \alpha^2 \hat{v}) - \alpha U_0'' \hat{v} = 0$$

(2nd order ODE)

Boundary conditions:

$$y = 0 \quad \hat{v} = 0$$
$$y = \infty \quad \hat{v} = 0$$

2 BCs

Examine when instability can occur in inviscid limit.

Assume temporal problem: $\alpha = \alpha_r$ given, $\omega = \omega_r + i\omega_i$

$$\left[ \hat{v}'' = \alpha_r^2 \hat{v} + \frac{\alpha_r U_0'' \hat{v}}{\alpha_r U_0 - \omega} \right] \hat{v}^*$$

$(\ )^*$ complex conj.

$$- \left[ \hat{v}^{*\prime\prime} = \alpha_r^2 \hat{v}^* + \frac{\alpha_r U_0'' \hat{v}^*}{\alpha_r U_0 - \omega^*} \right] \hat{v}$$

$$\rightarrow \quad \hat{v}'' \hat{v}^* - \hat{v}^{*\prime\prime} \hat{v} = \alpha_r U_0'' |\hat{v}|^2 \left\{ \frac{1}{\alpha_r U_0 - \omega} - \frac{1}{\alpha_r U_0 - \omega^*} \right\}.$$

$$\int_0^\infty \frac{d}{dy} \left( \hat{v}' \hat{v}^* - \hat{v}^{*\prime} \hat{v} \right) dy = \int_0^\infty \frac{\alpha_r U_0'' |\hat{v}|^2 \, 2i\omega_i \, dy}{|\alpha_r U_0 - \omega|^2}$$

$$0 = \omega_i \int_0^\infty \frac{U_0'' |\hat{v}|^2}{|U_0 - \omega/\alpha_r|^2} dy$$

Instability ($\omega_i > 0$, $|\hat{v}|^2 \neq 0$) possible only if $U_0''$ changes sign