5.1) Asymptotic Perturbation Theory

A) Basis for IBLT

B) 2D interaction models.

Reading: paper, handouts

Recap – BL imposes u_e at separation

A) Basis for IBLT

Recall from (3.2), we derived non-dimension form of N-S equations for 2-D steady, incompressible, viscous flow, and examined eqns. for $Re \to \infty$ → TSL equations

$$\vec{\nabla} \cdot \vec{u} = 0$$

$$\cancel{\frac{\partial \vec{u}}{\partial t}}^{0} + (\vec{u} \cdot \vec{\nabla})\vec{u} = -\frac{\vec{\nabla} p}{\rho} + \underbrace{\frac{1}{Re}}_{1/\epsilon^2} \cdot \nu \nabla^2 \vec{u} \qquad \text{(all * quant.)}$$

Recall from asymptotic analysis, we expand $\vec{u}$ in terms of ϵ

$$u = u_0 + \epsilon u_1 + \epsilon^2 u_2 + \cdots$$

$$v = v_0 + \epsilon v_1 + \epsilon^2 v_2 + \cdots$$

(asymptotic series)

$$p = p_0 + \epsilon p_1 + \epsilon^2 p_2$$

Rescaled $u, v \to U, V$ and $x, y \to X, Y$ (since ϵ multiplies $\nabla^2 \vec{u}$) singular perturbation)

$U = u$ $\quad X = x$

$V = v/\epsilon$ $\quad Y = y/\epsilon$ – stretched coordinate

$U = U_0 + \epsilon U_1 + \epsilon^2 U_2$, $V = V_0 + \epsilon V_1 + \cdots$

Substituting above: $\qquad \epsilon U_1 \cdot n_x + V_1$

Governing Eqns + Matching conditions

Outer problem

$$\nabla\cdot\vec{u} = 0$$

$$(\vec{u}\cdot\vec{\nabla})\vec{u} = -\nabla p + \epsilon^2\nabla^2\vec{u}$$

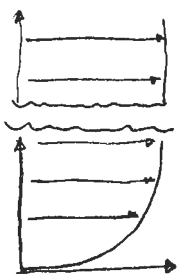
$$\left[u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \epsilon^2\frac{\partial^2 u}{\partial x^2} + \epsilon^2\frac{\partial^2 u}{\partial y^2} \right]$$

Inner Problem

$$\vec{\nabla}\cdot\vec{U} = \frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y} = 0$$

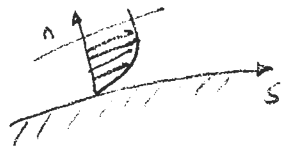
inner variables $\rightarrow$ $$U\frac{\partial U}{\partial X} + V\frac{\partial U}{\partial Y} = -\frac{\partial p}{\partial X} + \epsilon^2\frac{\partial^2 U}{\partial X^2} + \frac{\partial U}{\partial Y^2}$$

Matching Conditions



outer : $\vec{u}\cdot\hat{n} = \epsilon V$

inner $U = \vec{u}\cdot\vec{s} = u_e$



Zeroth order

$\vec{u} = \vec{u}_0$, $\vec{U} = \vec{U}_0(X,Y)$, drop all ϵ and higher

outer problem

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x}$$

inner problem

$$\frac{\partial U}{\partial X} + \frac{\partial V}{\partial y} = 0$$

$$U\frac{\partial U}{\partial X} + V\frac{\partial U}{\partial Y} = -\frac{\partial p}{\partial X} + \frac{\partial^2 U}{\partial Y^2}$$

matching cond: $\vec{u}\cdot\vec{n} = 0$ outer problem, $U = \vec{u}\cdot\vec{s} = u_e$ inner problem

⇒ Classical B·L formulation – potential flow + BL eqn (uncoupled) using u_e from potential flow

First order equations: $(\vec{u} = \vec{u}_0 + \epsilon \vec{u}_1 \quad , \quad \vec{V} = V_0 + \epsilon \vec{V}_1(X, Y))$

Outer problem: same

Inner problem: same

} as 0^{th} order.

Matching Cond.

appearance of Re

$$u \cdot \vec{n} = \epsilon V \quad \leftarrow \text{note}$$

$$U = \vec{u} \cdot \vec{s} = u_e$$

> IBLT → potential flow + BL equations, coupled matching conditions

<u>Iteration</u>

Classical (one-way coupled)

$$\begin{tikzcd}
\nabla^2 \phi = 0,\ \vec{\nabla}\phi \cdot \hat{n} = 0 \arrow[r] & u_e \arrow[d] \\
\tau_{w0} & B \cdot L \text{ eqns} \arrow[l]
\end{tikzcd}$$

IBLT (fully coupled)

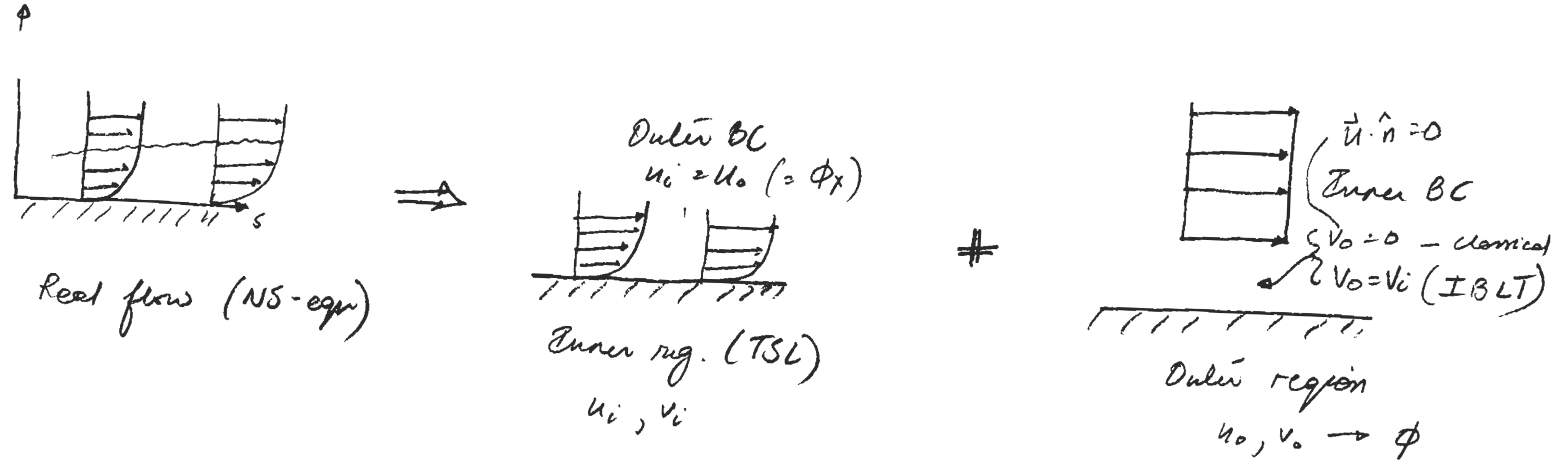
Attached ↻

Separated ↺

$$\begin{tikzcd}
\nabla^2 \phi = 0,\ \vec{\nabla}\phi \cdot \hat{n} = \epsilon V \arrow[r] & u_e \arrow[d] \\
V \arrow[u] & B \cdot L \text{ eqns } (\tau_w = \tau_{w0} + \epsilon \tau_{w1}) \arrow[l]
\end{tikzcd}$$

Forward (simple) iterations fails for IBLT when flow separates

Classical vs IBLT



Compare Classical vs IBLT

Classical

- $v_o = 0$
- Outer region decouples from inner (can be solved invid.)
- "Correct" in limit of $Re \to \infty$
  error in $v_i \approx O(1/\sqrt{Re})$

  <u>Except if:</u>
  - Separation occurs
    error in $v_i \approx O(1)$
  - Drag calculation for attached or separated flow
    drag $= O(1/\sqrt{Re}) \approx$ error

IBLT

$v_o = v_i = O(1/\sqrt{Re})$
(attached)

- Outer & inner regions coupled (must be solved together)
- Correct in limit of $Re \to \infty$
- Particularly OK if
  - limited separation
    $\frac{d\delta^*}{dx} << 1 \approx 0.1$
  - drag is to be calculated
- more accurate for large Re but less than $Re \to \infty$ case.

<u>outer</u> $\quad u = \underbrace{u_0}_{\text{class}} + \underbrace{\epsilon u_1}_{\text{IBLT}} + \epsilon^2 u_2$

Inner $\quad U = \underbrace{U^0}_{\text{std TSL}} + U_1^* \epsilon + \cdots$

U_1^* — higher order BL theory.

$$\vec{V} = \vec{U}_0 + \epsilon \vec{U}_1$$

$$\vec{V} \cdot \hat{n} = \cancel{\vec{U}_0 \cdot \hat{n}}^{\,0} + \epsilon \vec{U}_1 \cdot \hat{n}$$

$$= \epsilon \left[U_1 n_x + V_1 n_y \right]$$

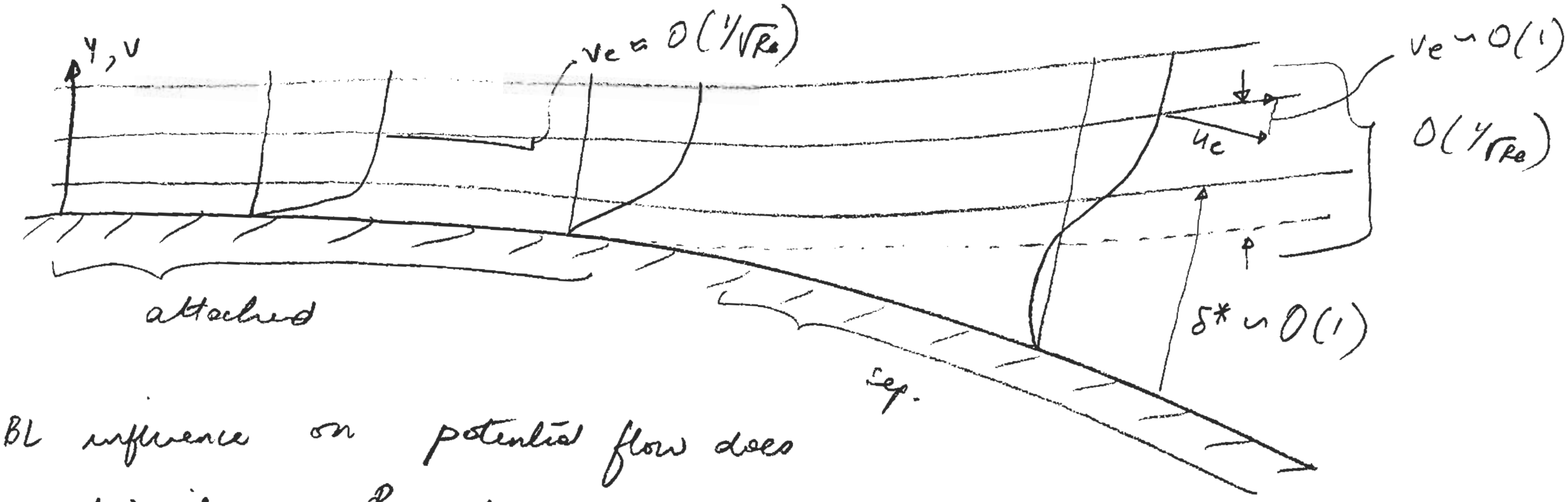
$$\left[u_1 n_x + \frac{v_1}{\epsilon} n_y \right]$$

$$= \epsilon u_1 n_x + v_1 n_y$$

$$= v_1 n_y$$

$$= \epsilon V_1 n_y$$

'assical BC ($V_0 = 0$) fails in separated flow



BL influence on potential flow does not diminish as $Re \to \infty$ in sep. flow

Cannot solve classical IB eqn past separation anyway.

> Displacement effects – 2D interaction model

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\int_0^{y_e} (\quad) \, dy = 0$$

$$\Rightarrow \quad V_e - V_0 = -\int_0^{y_e} \frac{\partial u}{\partial x} \, dy$$

$$= -\left[\frac{\partial}{\partial x}\int_0^{y_e} u \, dy - u(y_e)\frac{dy_e}{dx} + u(0)\frac{dy_0}{dx}\right]$$

$$= -\frac{d}{dx}\left[(\delta - \delta^*) u_e\right] + u_e \frac{dy_e}{dx}$$

$$= -u_e \cancel{\frac{d\delta}{dx}} - \delta \frac{du_e}{dx} + \frac{du_e}{dx}\delta^* + \cancel{u_e \frac{dy_e}{dx}}$$

$$V_e - V_0 = \frac{d}{dx}(u_e \delta^*) - \delta \frac{du_e}{dx}$$

Leibnitz

$$\frac{\partial}{\partial x}\int_{\phi_1(x)}^{\phi_2(x)} f(x,y)\, dy$$

$$= \int_{\phi_1}^{\phi_2} \frac{\partial}{\partial x} f(x,y)\, dy + f(x, \phi_2)\frac{d\phi_2}{dx} - f(x, \phi_1)\frac{d\phi_1}{dx}$$

B) Implications for drag and lift predictions

Basis for Interacting Boundary Layer Theory

Solve viscous flow equations via asymptotic series in the small parameter $\epsilon \equiv Re^{-1/2}$

$$u(x,y,\epsilon) = u_0(x,y) + \epsilon u_1(x,y) + \epsilon^2 u_2(x,y) + \ldots \qquad \nabla \cdot \vec{u} = 0$$
$$v(\quad) = v_0(\quad) + \epsilon v_1(\quad) + \epsilon^2 v_2(\quad) + \ldots$$
$$p(\quad) = p_0(\quad) + \epsilon p_1(\quad) + \epsilon^2 p_2(\quad) + \ldots \qquad \vec{u} \cdot \nabla \vec{u} = -\nabla p + \epsilon^2 \nabla^2 \vec{u}$$

Since ϵ multiplies highest-order derivative $\nabla^2 \vec{u}$, this is a singular perturbation. Must use separate rescaled variables near wall.

$$U(X,Y,\epsilon) = U_0(X,Y) + \epsilon U_1(X,Y) + \ldots \qquad X = x \quad Y = y/\epsilon$$
$$V(\quad) = V_0(\quad) + \epsilon V_1(\quad) + \ldots \qquad U = u \quad V = v/\epsilon$$

Now: $\epsilon^2 \nabla^2 u = \frac{\partial^2 U}{\partial Y^2} + \mathcal{O}(\epsilon$

Governing equations and matching conditions at δ

Outer problem:

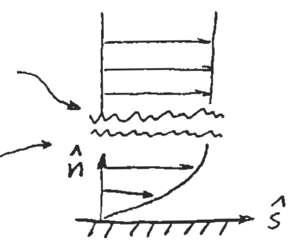
$$\nabla \cdot \vec{u} = 0$$
$$\vec{u} \cdot \nabla \vec{u} = -\nabla p + \epsilon^2 \nabla^2 \vec{u}$$

$\vec{u} \cdot \hat{n} = \epsilon V$

Inner problem:

$$\frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y} = 0$$
$$U \frac{\partial U}{\partial X} + V \frac{\partial U}{\partial Y} = -\frac{\partial P}{\partial X} + \epsilon^2 \frac{\partial^2 U}{\partial X^2} + \frac{\partial^2 U}{\partial Y^2}$$

$U = \vec{u} \cdot \hat{s}$



Zeroth-Order Equations: $\vec{u} = \vec{u}_0 \quad \vec{U} = \vec{U}_0$

$$\nabla \cdot \vec{u} = 0$$
$$\vec{u} \cdot \nabla \vec{u} = -\nabla p$$

$u \cdot \hat{n} = 0$

$$\frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y} = 0$$
$$U \frac{\partial U}{\partial X} + V \frac{\partial U}{\partial Y} = u_e \frac{du_e}{dX} + \frac{\partial^2 U}{\partial Y^2}$$

$U = u_e$

First-Order Equations: $\vec{u} = \vec{u}_0 + \epsilon \vec{u}_1 \ , \quad \vec{U} = \vec{U}_0 + \epsilon \vec{U}_1$

$$\nabla \cdot \vec{u} = 0$$
$$\vec{u} \cdot \nabla \vec{u} = -\nabla p$$

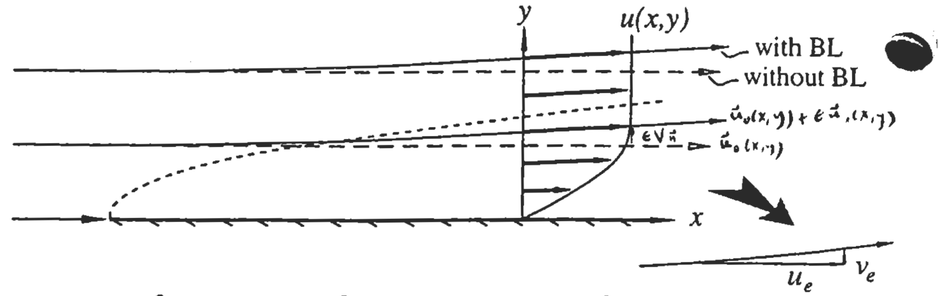
$\vec{u} \cdot \hat{n} = \epsilon V$

$$\frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y} = 0$$
$$U \frac{\partial U}{\partial X} + V \frac{\partial U}{\partial Y} = u_e \frac{du_e}{dX} + \frac{\partial^2 U}{\partial Y^2}$$

$U = u_e$

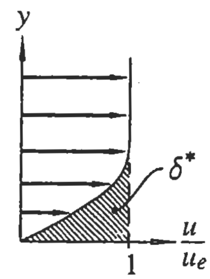
Displacement Effects of Boundary Layer on Potential Flow

Actual Flow



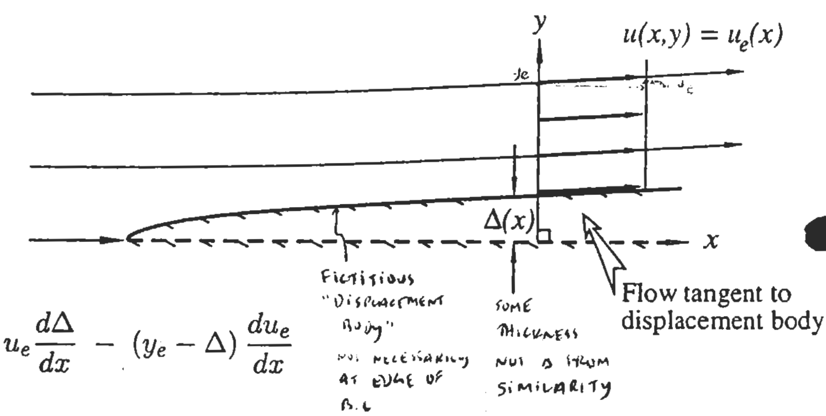
$$v(x,y_e) \equiv v_e(x) = \int_0^{y_e} \frac{\partial v}{\partial y}\, dy = -\int_0^{y_e} \frac{\partial u}{\partial x}\, dy = \int_0^{y_e} \frac{\partial}{\partial x}(u_e - u)\, dy - y_e \frac{du_e}{dx}$$

$$= \frac{d}{dx}\left[u_e \int_0^{y_e}\left(1 - \frac{u}{u_e}\right) dy\right] - y_e \frac{du_e}{dx}$$



or $\boxed{v_e = \frac{d}{dx}(u_e \delta^*) - y_e \frac{du_e}{dx}}$ where $\delta^* = \int_0^{y_e}\left(1 - \frac{u}{u_e}\right) dy$

Displacement Body Model

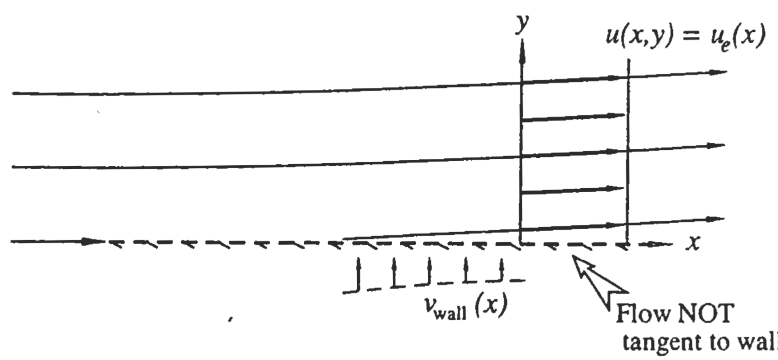


$$v_e(x) = u_e \frac{d\Delta}{dx} + \int_\Delta^{y_e} \frac{\partial v}{\partial y}\, dy$$

$$= u_e \frac{d\Delta}{dx} - \int_\Delta^{y_e} \frac{\partial u}{\partial x}\, dy = u_e \frac{d\Delta}{dx} - (y_e - \Delta)\frac{du_e}{dx}$$

or $\boxed{v_e = \frac{d}{dx}(u_e \Delta) - y_e \frac{du_e}{dx}}$ $\Longrightarrow$ $\underline{\underline{\Delta = \delta^*}}$ (by comparing with Actual Flow v_e)

Wall Blowing Model



$$v_e(x) = v_{\text{wall}} + \int_0^{y_e} \frac{\partial v}{\partial y}\, dy$$

$$= v_{\text{wall}} - \int_0^{y_e} \frac{\partial u}{\partial x}\, dy$$

or $\boxed{v_e = v_{\text{wall}} - y_e \frac{du_e}{dx}}$ $\Longrightarrow$ $\underline{\underline{v_{\text{wall}} = \frac{d}{dx}(u_e \delta^*)}}$ (by comparing with Actual Flow)