Seminar in Perturbation Methods: Formation of Shock Waves

1 References

Moran and Shen (1996), JFM, Vol. 25, pp. 705-718.

Mason (1968), Phys. Fluids, Vol. II, pp. 2524-2532.

Moran (1965), Cornell University thesis.

2 Background and Introduction

In the absense of friction and heat conduction (small gradients in velocity and temperature are assumed), an isentropic disturbance will propagate without change in shape. The particle velocity under these conditions is governed by the wave equation:

$$\frac{\partial^2 u}{\partial t^2} = a_1^2 \frac{\partial^2 u}{\partial x^2} \tag{2.1}$$

whose solution may be expressed as:

$$u(x,t) = F(x - a_1 t) + G(x + a_1 t) \tag{2.2}$$

The wave propagates at the local speed of sound a_1, where a_1 is a constant.

An acoustic wave of finite amplitude will propagate with shape being distorted in the process. The theoretical description is expressed in terms of non-linear equations. The wave speed for acoustic waves of finite amplitude is

$$c = a_1 \pm \left(\frac{\gamma+1}{2}\right) u \tag{2.3}$$

$$c = a_1 \left\{ 1 \pm \frac{\gamma+1}{\gamma-1} \left[\left(\frac{\rho}{\rho_1}\right)^{\frac{\gamma-1}{2}} - 1 \right] \right\} \tag{2.4}$$

Thus, the regions of higher condensation, $(\frac{\rho}{\rho_1}) > 1$, tend to overtake those of lower condensation. This non-linear steepening effect resulting from the convective terms of the equations of motion is eventually balanced by the diffusive terms. The diffusive terms become important as the velocity and temperature gradients become larger (steepening effect). When complete balance is obtained, the wave is then "stationary" and propagates without further distortion.

Thus, for acoustic waves of finite amplitude one may anticipate two time scales. Physically the time scales would correspond to (a) the time in which the non-linear convective terms determine the nature of the wave propagation and (b) the time in which the diffusive terms are of the same order as the non-linear convective terms.

3 The linearized solution

We shall discuss the formation of weak plane shock waves by impulsive motion of a piston as treated by Moran and Shen (JFM, 1966). In this problem we also have two time scales. After the piston is set in motion the wave propagation is governed by the viscous diffusive terms resulting from the steep gradients in flow initially. This part of the wave propagation constitutes a time scale. The second time scale is characterized by the non-linear convective terms balancing the viscous diffusive terms, the initial gradients being smaller on this scale.

Using a continuum flow formulation Moran and Shen consider the phenomenon for time large compared with the mean time spent by a gas molecule between collisions. Denoting the piston Mach number by ϵ the linearized Navier-Stokes equations may be shown to be valid up to time of the order of $1/\epsilon^2$ mean free times after the piston is set in motion. At large times the solution may be shown to be governed by Burgers's equation. Boundary conditions for the large time solution are obtained by applying the matching principle of the method of inner and outer expansions. In their analysis Moran and Shen assumed ϵ to be small; the gas is viscous and heat conducting.

The basic equations and non-dimensionalization are:

$$\mu^* = \epsilon\sqrt{RT_0^*}\mu, \quad \rho^* = \rho_0^*(1+\epsilon\rho) \tag{3.1}$$

$$p^* = p_0^*(1+\epsilon p), \quad T^* = T_0^*(1+\epsilon T) \tag{3.2}$$

where $(\quad)^*$ denotes a dimensional variable, R is the gas constant. ϵ is a perturbation parameter, defined so that $\mu = 1$ on the piston. ϵ is proportional to the piston Mach number. $(\quad)_0$ enotes the undisturbed value of the variable.

The viscosity is given by

$$\mu^* = \mu_0^*(1+\epsilon\mu) \tag{3.3}$$

The dimensionless independent variables x and t are given by

$$x^* = \left\{\mu_0^*/\rho_0^*\sqrt{RT_0^*}\right\}x \tag{3.4}$$

$$t^* = \left\{\mu_0^*/\rho_0^* RT_0^*\right\}t \tag{3.5}$$

The exact Navier-Stokes conservation equations in a one dimensional unsteady flow may be written:

$$\rho_t + \mu_x + \epsilon(\rho\mu)_x = 0 \tag{3.6}$$

$$\mu_t + p_x - \mu_{xx} + \epsilon\left[\rho\mu_t + \mu\mu_x - (\mu\mu_x)_x\right] + \epsilon^2\rho\mu\mu_x = 0 \tag{3.7}$$

$$T_t+(\gamma-1)\mu_x-\frac{\gamma}{\nabla}T_{xx}+\epsilon\Big[\rho T_t+\mu T_x+(\gamma-1)p\mu_x-(\gamma-1)\mu_x^2-\frac{\gamma}{\nabla}(\mu T_x)_x\Big]+\epsilon^2\Big[\rho\mu T_x+(\gamma-1)\mu\mu_x^2\Big]=0 \quad (3.8)$$

$$p=\rho+T+\epsilon\rho T \quad (3.9)$$

where

$$\gamma = \text{specific heat ratio}$$

$$\nabla = \text{Prandtl number (based on the longitudinal viscosity)}$$

The initial conditions for the piston problem are:

$$\mu=\rho=p=T=0, \quad x>0, \quad t=0 \quad (3.10)$$

For an impermeable and adiabatic piston, the boundary conditions are (at the piston)

$$\mu=1, \quad T_x=0, \quad x=\epsilon t, \quad t>0 \quad (3.11)$$

At infinity, the damping conditions are imposed

$$\mu,\rho,T\longrightarrow 0, \quad t>0, \quad x\to\infty \quad (3.12)$$

The linearized solution is obtained by letting $\epsilon\to\infty$. As expected from the physics of the problem, by letting $\epsilon\to\infty$ the non-linear convective terms are lost and the first order viscous diffusive terms are retained. Our equation system reduces to the following linear set of equations:

$$\rho_t^o+\mu_x^o=0 \quad (3.13)$$

$$\mu_t^o+p_x^o-\mu_{xx}^o=0 \quad (3.14)$$

$$T_t^o+(\gamma-1)\mu_x^o-\frac{\gamma}{\nabla}T_{xx}^o=0 \quad (3.15)$$

$$p^o=\rho^o+T^o \quad (3.16)$$

The initial conditions and boundary conditions reduce to:

$$\mu^o=\rho^o=T^o=0, \quad x>0, \quad t=0 \quad (3.17)$$

$$\mu^o=1, \quad T_x^o=0, \quad x=0, \quad t>0 \quad (3.18)$$

$$\mu^o,\rho^o,T^o\longrightarrow 0, \quad t>0, \quad x\to\infty \quad (3.19)$$

In equations (3.13) to (3.19), the superscript o denotes the linearized solution or outer variables.

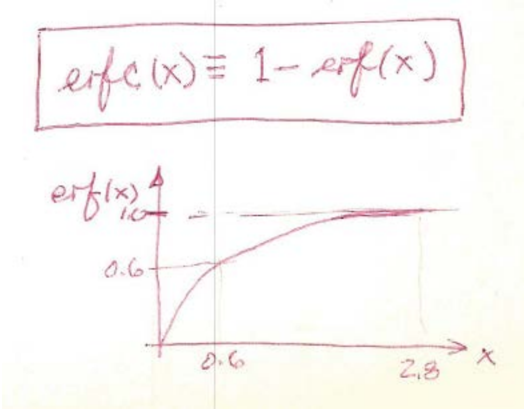
We solve for the linearized dependent variables by using Laplace transforms:

$$\overline{\mu^o}(x,s)=\int_0^\infty e^{-st}\mu^o(x,t)\,dt \quad (3.20)$$

The general formulae for the asymptotic behavior of the solution (real world) are

$$\begin{cases}\mu^o(x,t) & \sim & \frac{1}{2}\mathrm{erfc}\Big[(x-\sqrt{\gamma}t)/\sqrt{2\beta t}\Big]+o(t^{-\frac{1}{2}})\\ \rho^o(x,t) & \sim & \frac{1}{\sqrt{\gamma}}\mu^o(x,t)+o(t^{-\frac{1}{2}})\\ T^o(x,t) & \sim & \frac{(\gamma-1)}{\sqrt{\gamma}}\mu^o(x,t)+o(t^{-\frac{1}{2}})\end{cases} \quad (3.21)$$

$$\beta\equiv 1+\frac{\gamma-1}{\nabla} \quad (3.22)$$



The linearized solution indicate a shock-like behavior at large times, i.e. the flow properties eventually exhibit a smooth transition between differing constant values. Also, the center of the transition propagates at the speed of sound (local) and the Pankine-Hugoniot relations are satisfied.

Let

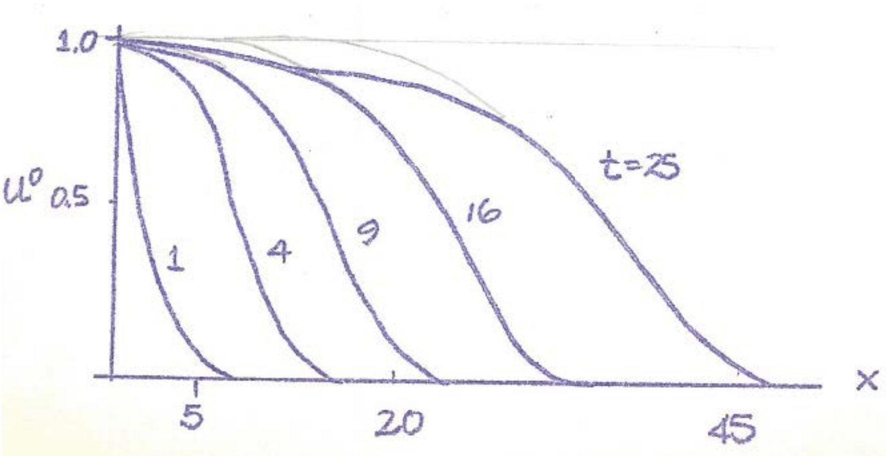
$$X = (x - \sqrt{\gamma}t)/\sqrt{2\beta} \tag{3.23}$$

The asymptotic behavior of the linearized solution may be written:

$$\begin{cases} \mu^o(x,t) & \sim & \frac{1}{2}\text{erfc}\left[X/\sqrt{t}\right] + o(t^{-\frac{1}{2}}) \\ \rho^o(x,t) & \sim & \frac{1}{\sqrt{\gamma}}\mu^o(x,t) + o(t^{-\frac{1}{2}}) \\ T^o(x,t) & \sim & \frac{(\gamma-1)}{\sqrt{\gamma}}\mu^o(x,t) + o(t^{-\frac{1}{2}}) \end{cases} \tag{3.24}$$

We also note that the relation between $\mu^o, \rho^o, and T^o$ is the same as that obtained by solving the propagation of isentropic, infinitesemal amplitude wave problem (the one-dimensional wave equation).

From equation (3.24) we note that the width of the transition zones grows in time like $\sqrt{t}$. The width of a weak shock in steady flow, in the above notation, is of order of $1/\epsilon$. Physically, we expect the solution of the piston problem to yield a steady (convective and diffusive terms enter to the same order) travelling shock as $t \to \infty$. This suggests that the linearized solution will break down when $\sqrt{t} = o\frac{1}{\epsilon}$ or $t = o(\frac{1}{\epsilon^2})$. A more formal argument leading to the above result is given in Moran (1966).



4 The solution at large times

We now define an "inner region" where t is of the same order or greater than $1/\epsilon^2$. In the inner regions the "inner variables" and their derivatives are of order unity in the limit $\epsilon \to 0$. Since the outer variables for large time are of order unity, they need not be rescaled or stretched. The independent variables must be stretched:

$$\tau = \epsilon^2 t \tag{4.1}$$

To rescale the distance we note that for large time, i.e. inside the inner region, the outer solution may be approximated by their asymptotic expansions. This means that outside the "shock" everything is constant and our interest should be directed toward the shock interior. Hence as $t \to \infty$ or $\tau \to \infty$, we center the inner distance coordinate with the shock and stretch it by ϵ to make the shock thickness on the inner scale to be order unity. Hence

$$\xi = \epsilon(x - \sqrt{\gamma}t) = \epsilon\sqrt{2\beta}X \tag{4.2}$$

Substituting equations (4.1) and (4.2) into Navier-Stokes equations one obtains a redundant system of equations to first order in ϵ. The redundancy is a result of stretching the x-coordinate as to make the inner equations inviscid to first order. Expanding the inner dependent variables in a power series in ϵ – e.g.

$$\mu^i = \mu_0^i + \epsilon\mu_1^i + \epsilon^2\mu_2^i + ... \tag{4.3}$$

and substituting equations (4.1), (4.2), and (4.3) into the Navier-Stokes equations, we obtain

$$\mu_\tau^i + \frac{1}{2}(\gamma+1)\mu^i\mu_\xi^i = \frac{1}{2}\beta\mu_{\xi\xi}^i \tag{4.4}$$

(Viscous dissipation - no dispersion)

Equation (4.4) is Burgers's equation.

The boundary conditions for equation (4.4) is obtained by matching the inner and outer solutions:

$$(\mu^i)^o = (\mu^o)^i \tag{4.5}$$

The inner solution is re-expressed in outer variables and re-expanded in ϵ for fixed x and t, the result ought to be the same as if the outer solution were put in inner variables and re-expanded for fixed ξ and τ.

We obtain as initial conditions for μ^i:

$$\begin{cases} \mu_i(\xi,0) = 0, & \xi > 0 \\ \mu^i(\xi,0) = 1, & \xi < 0 \end{cases} \tag{4.6}$$

Hence on the inner scale we have an initial value problem. First we transform μ^i:

$$\mu^i = -\frac{2\beta}{(\gamma+1)}\frac{\Psi_\xi}{\Psi} \tag{4.7}$$

Combining equations (4.4), (4.6), and (4.7) and integrating over ξ, we find

$$\Psi_\xi = \frac{1}{2}\beta\Psi_{\xi\xi} \tag{4.8}$$

$$\begin{cases} \Psi(\xi,0) = \exp(-(\gamma+1)\xi/2\beta), & \xi < 0 \\ \Psi(\xi,0) = 1, & \xi \geq 0 \end{cases} \tag{4.9}$$

Equation (4.8) is the heat equation. Solution can be obtained by disturbing heat sources along the ξ-axis. See Moran (1966). The inner solution when matched to the outer solution shows that the asymptotic matching principle is satisfied (as opposed to the limit matching principle).

The composite solution is given by

$$\begin{cases} \mu_c = \mu^i + \mu^o - (\mu^i)^o \\ \mu^c = \mu^i \mu^o / (\mu_o)^i \end{cases} \tag{4.10}$$

$$u^i(\xi,\tau) = \left[1 + \frac{exp[(\gamma+1)/2\beta(\xi - \frac{1}{4}(\gamma+1)\tau)]erfc[\xi/\sqrt{2\beta\tau}]}{erfc[(\xi - \frac{1}{2}(\gamma+1)\tau)/\sqrt{2\beta\tau}]} \right]^{-1} \tag{4.11}$$