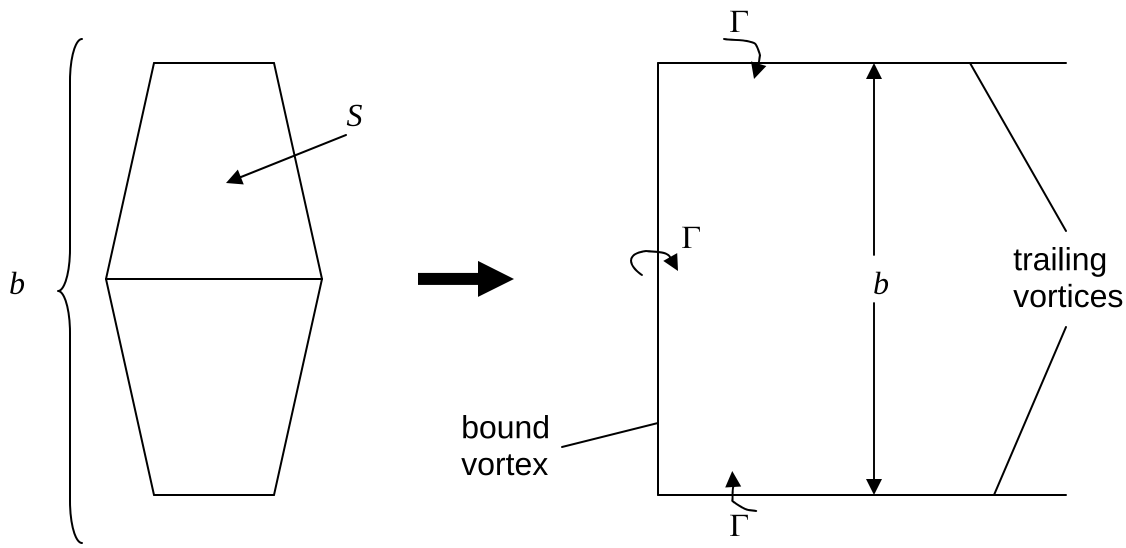
Single Horseshoe Vortex Wing Model



<u>Lift due to a horseshoe vortex</u>

Kutta-Joukowsky Theorem

$$L = \rho_\infty V_\infty \int_{-\frac{b}{2}}^{\frac{b}{2}} \Gamma dy = \rho_\infty V_\infty \Gamma b$$

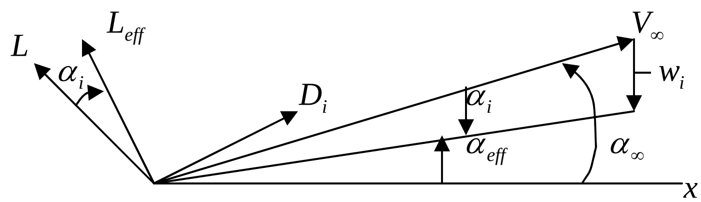
$$C_L = \frac{L}{\frac{1}{2}\rho_\infty V_\infty^2 S} = \frac{\rho_\infty V_\infty \Gamma b}{\frac{1}{2}\rho_\infty V_\infty^2 S}$$

$$C_L = \frac{2\Gamma}{V_\infty b}\frac{b^2}{S}$$

$$C_L = \frac{2\Gamma}{V_\infty b} A$$

<u>Induced Drag</u>

To estimate the induced drag using this simple model, we will assume that the 3-D lift is tilted by the downwash occurring at the wing root ($y = 0$).



$L \perp V_\infty$

$L \perp D_i$

$$\alpha_{eff} = \alpha_\infty - \alpha_i$$

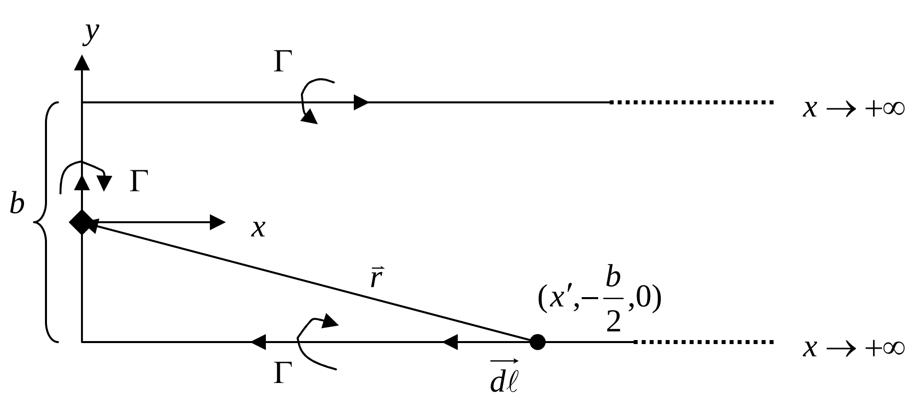
$$L = L_{eff} \cos\alpha_i \approx L_{eff}$$

$$D_i = L_{eff} \sin\alpha_i \approx L_{eff}\alpha_i = L\alpha_i$$

For small $\alpha_\infty \, \& \, \alpha_i$

$$\alpha_i \approx \frac{-w_i}{V_\infty}$$

To calculate downwash, we apply Biot-Savart:



$$\vec{w}(x,y,z) = \frac{\Gamma}{4\pi} \int_{filament} \frac{d\vec{\ell} \times \vec{r}}{r^3}$$

At $x = y = z = 0$ (i.e. wire root), the bound vortex does not produce a downwash, so, we only have the 2 trailing vortices at $y = \pm b/2$.

$$\vec{w}(0,0,0) = \frac{\Gamma}{4\pi}\left[\int_{y=-\frac{b}{2}} + \int_{y=+\frac{b}{2}} \right]$$

Let's do the $y = -\frac{b}{2}$ integral first:

$$\int_{x=+\infty}^{x=0} \frac{d\vec{\ell} \times \vec{r}}{r^3} = \int_{x=+\infty}^{0} \frac{(-dx'\vec{i}\,) \times \left[(-x')^2\,\vec{i} + (+\frac{b}{2})^2\,\vec{j} \right]}{\left[x'^2 + (\frac{b}{2})^2 \right]^{\frac{3}{2}}}$$

$$= \int_{+\infty}^{0} \frac{-\frac{b}{2}dx'\vec{k}}{\left[x'^2 + (\frac{b}{2})^2 \right]^{\frac{3}{2}}}$$

$$\boxed{w(0,0,0) = -\frac{\Gamma}{\pi b}}$$

Then, combining this we can find:

$$D_i \cong L\alpha_i = -L\frac{w_i}{V_\infty}$$

$$D_i = -L(\frac{-\Gamma}{\pi b V_\infty})$$

$$D_i = \frac{L\Gamma}{\pi b V_\infty}$$

But $\quad L = \rho_\infty V_\infty \Gamma b \Rightarrow \Gamma = \frac{L}{\rho_\infty V_\infty b}$

$$\Rightarrow \boxed{D_i = \frac{L^2}{\pi \rho_\infty V_\infty^2 b^2}}$$

$$\Rightarrow C_{D_i} = \frac{D_i}{\frac{1}{2}\rho_\infty V_\infty^2 S} = \left(\frac{L}{\frac{1}{2}\rho_\infty V_\infty^2 S}\right)^2 \frac{S}{2\pi b^2}$$

$$\boxed{C_{D_i} = \frac{C_L^2}{2\pi A}}$$

Hmmm is this ok??

Recall: Lifting line:

$$C_{D_i} = \frac{C_L^2}{\pi A e}, e \le 1$$