- Assume steady

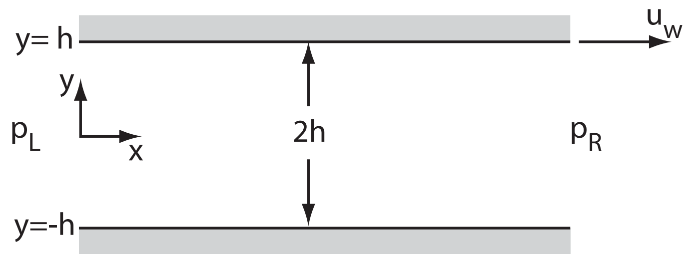
$$\Rightarrow \frac{\partial}{\partial t} = 0$$

- Assume $\frac{L}{h} >> 1$

$$\Rightarrow \frac{\partial \vec{V}}{\partial x} = 0$$

- Assume 2-D

$$\Rightarrow w = 0, \frac{\partial}{\partial z} = 0$$



<u>Incompressible N-S equations:</u>

1. $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$
2. $\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$
3. $\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)$

<u>BC's</u>

$$v(x, \pm h) = 0$$
$$u(x, -h) = 0$$
$$u(x, +h) = u_w$$

Turning the crank:

$$\underbrace{\frac{\partial u}{\partial x}}_{\frac{\partial}{\partial x}=0} + \frac{\partial v}{\partial y} = 0 \Rightarrow \frac{\partial v}{\partial y} = 0 \Rightarrow v = v(x)$$

$$\text{but } \frac{\partial v}{\partial x} = 0 \Rightarrow v = \text{const}$$

$$\text{Apply bc's} \quad \Rightarrow \quad v = 0$$

Now, $y-$momentum : Since $v = 0$, we have:

$$\frac{\partial p}{\partial y} = 0 \Rightarrow p(x, y) = p(x)$$

Note: since the pressure does change from p_L to p_R over the length $L, p = p(x)$.

Finally $x-$momentum :

$$\underbrace{\frac{\partial u}{\partial t}}_{\substack{steady\\ \frac{\partial}{\partial t}=0}} + u\underbrace{\frac{\partial u}{\partial x}}_{=0} + \underbrace{v}_{=0}\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{dp}{dx} + \nu\left(\underbrace{\frac{\partial^2 u}{\partial x^2}}_{\frac{\partial}{\partial x}=0} + \frac{\partial^2 u}{\partial y^2}\right)$$

$$\Rightarrow \frac{\partial^2 u}{\partial y^2} = -\frac{1}{\mu}\frac{dp}{dx}, \text{where } \nu \equiv \frac{\mu}{\rho}$$

Observe that $LHS = f(y)$ and $RHS = g(x)$

$$\Rightarrow f(y) = g(x) = \text{const.}$$

$$\Rightarrow \frac{dp}{dx} = \text{const} = \frac{p_R - p_L}{L}$$

For this problem, I'll just use the gradient $\frac{dp}{dx}$ but realize this is specified by the end pressures.

Next, integrate in y:

$$\int\left[\frac{d^2 u}{dy^2} = -\frac{1}{\mu}\frac{dp}{dx}\right]dy$$

$$\Rightarrow \int\left[\frac{du}{dy} = -\frac{1}{\mu}\frac{dp}{dx}y + C_1\right]dy$$

$$\Rightarrow u = -\frac{1}{2\mu}\frac{dp}{dx}y^2 + C_1 y + C_o$$

Now, apply bc's:

$$u(y=-h) = -\frac{1}{2\mu}\frac{dp}{dx}h^2 - C_1 h + C_o = 0$$

$$u(y=+h) = -\frac{1}{2\mu}\frac{dp}{dx}h^2 + C_1 h + C_o = u_w$$

Solving for C_o & C_1 :

$$C_o = \frac{1}{2}\left(u_w + \frac{1}{\mu}\frac{dp}{dx}h^2\right)$$

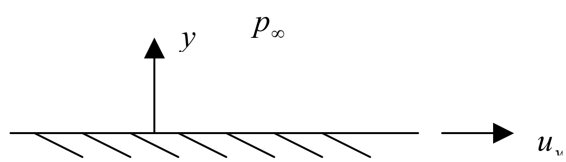
$$C_1 = \frac{u_w}{2h}$$

$$\Rightarrow \boxed{u(y) = -\frac{h^2}{2\mu}\frac{dp}{dx}\left[\left(\frac{y}{h}\right)^2 - 1\right] + \frac{u_w}{2}\left(\frac{y}{h}+1\right)}$$

Suddenly started flat plate (Stokes 1st Problem)

IC: $t = 0, \begin{cases} u = 0 \\ v = 0 \end{cases}$

BC: $t > 0, \begin{cases} u(x,0) = u_w \\ v(x,0) = 0 \end{cases}$



Assume infinite length, $\frac{\partial}{\partial x} = 0$

<u>Continuity:</u>

$$\underbrace{\frac{\partial u}{\partial x}}_{=0} + \frac{\partial v}{\partial y} = 0 \Rightarrow v = v(x)$$

$$\text{but } \frac{\partial v}{\partial x} = 0 \text{ so } v = 0$$

$y-$momentum:

$$\underbrace{\frac{\partial v}{\partial t}}_{=0} + u\underbrace{\frac{\partial v}{\partial x}}_{=0} + v\underbrace{\frac{\partial v}{\partial y}}_{=0} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\underbrace{\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)}_{=0}$$

$$\frac{\partial p}{\partial y} = 0 \Rightarrow p = p(x) = p_\infty$$

$x-$momentum :

$$\frac{\partial u}{\partial t}+u\underbrace{\frac{\partial u}{\partial x}}_{\frac{\partial}{\partial x}=0}+\underbrace{v}_{=0}\frac{\partial u}{\partial y}=-\frac{1}{\rho}\underbrace{\frac{\partial p}{\partial y}}_{p=p_\infty}+\nu\left(\underbrace{\frac{\partial^2 u}{\partial x^2}}_{\frac{\partial}{\partial x}=0}+\frac{\partial^2 u}{\partial y^2}\right)$$

$$\Rightarrow \frac{\partial u}{\partial t}=\nu\frac{\partial^2 u}{\partial y^2}$$

This is the diffusion equation (also known as heat equation).

- There are many ways to solve this equation
- We'll use a similarity solution approach used in boundary layer theory.

<u>Similarity Solution</u>

- Assume that $u(t,y)=u(\eta)$ where $\eta=\eta(t,y)$. Reduce PDE to ODE.
- Usually, the assumption is made that:

$$\eta = Ct^a y^b$$

$$\Rightarrow \frac{\partial \eta}{\partial t}=aCt^{a-1}y^b=\frac{a\eta}{t}$$

$$\frac{\partial \eta}{\partial y}=bCt^a y^{b-1}=\frac{b\eta}{y}$$

$$\Rightarrow \boxed{\frac{\partial u}{\partial t}=\frac{du}{d\eta}\frac{\partial \eta}{\partial t}=\frac{a\eta}{t}\frac{du}{d\eta}}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial y}\left[\frac{du}{d\eta}\frac{d\eta}{dy}\right]$$

$$= \frac{\partial}{\partial y}\left[\frac{du}{d\eta}\frac{b\eta}{y}\right]$$

$$= \frac{b\eta}{y}\frac{\partial}{\partial y}\left(\frac{du}{d\eta}\right) + \frac{du}{d\eta}\frac{\partial}{\partial y}\left(\frac{b\eta}{y}\right)$$

$$= \left(\frac{b\eta}{y}\right)^2 \frac{\partial^2 u}{\partial \eta^2} + \frac{du}{d\eta}\frac{\partial}{\partial y}(bCt^a y^{b-1})$$

$$= \left(\frac{b\eta}{y}\right)^2 \frac{\partial^2 u}{\partial \eta^2} + \frac{du}{d\eta} b(b-1)Ct^a y^{b-2}$$

$$\boxed{\frac{\partial^2 u}{\partial y^2} = (\frac{b\eta}{y})^2 \frac{d^2 u}{d\eta^2} + b(b-1)\frac{\eta}{y^2}\frac{du}{d\eta}}$$

Thus:

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2} \quad \text{becomes}$$

$$\frac{a\eta}{t}\frac{du}{d\eta} = \nu\left(\frac{b\eta}{y}\right)^2 \frac{d^2 u}{d\eta^2} + \nu b(b-1)\frac{\eta}{y^2}\frac{du}{d\eta}$$

Re-arranging:

$$\frac{d^2 u}{d\eta^2} = \left[\frac{a}{\nu b^2}\underbrace{\frac{y^2}{t\eta}}_{\substack{\text{needs to be}\\ f(\eta)\text{ only}}} - \frac{b-1}{b\eta}\right]\frac{du}{d\eta}$$

$$\Rightarrow \eta = C\left(\frac{y}{\sqrt{t}}\right)^b$$

For simplicity, $b = 1$ and $C = \frac{1}{2\sqrt{\nu}}$

$$\Rightarrow a = -\frac{1}{2}$$

$$\Rightarrow \eta = \frac{y}{\sqrt{\nu t}}$$

$$\Rightarrow \boxed{\frac{d^2u}{d\eta^2} = -2\eta\frac{du}{d\eta}}$$

Note: bc is $u(0) = u_w$
$u(\eta \to \infty) = 0 \leftarrow$ Also is correct initial condition

$$\frac{du}{d\eta} = Ce^{-\eta^2}$$

Integrate again

$$u(\eta) = C\int_0^\eta e^{-\beta^2}d\beta + C_o$$