Effect of Turbulent Fluctuations on Mean Flow: Reynolds-Averaging

In a turbulent flow, we can define the mean, steady flow as:

$$\overline{u}(x,y,z) = \lim_{T\to\infty} \frac{1}{T}\int_0^T u(x,y,z,t)\,dt$$

This allows us to split the flow properties into a mean and a fluctuating part:

$$\begin{aligned}
u(x,y,z,t) &= \overline{u}(x,y,z) + u'(x,y,z,t)\\
v(x,y,z,t) &= \overline{v}(x,y,z) + v'(x,y,z,t)\\
w(x,y,z,t) &= \overline{w}(x,y,z) + w'(x,y,z,t)\\
p(x,y,z,t) &= \underbrace{\overline{p}(x,y,z)}_{\text{mean part}} + \underbrace{p'(x,y,z,t)}_{\text{turbulent fluctuating part}}
\end{aligned}$$

Note: the mean of u' is zero:

$$u - \overline{u} = u'$$

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T \{u - \overline{u}\}\,dt = \lim_{T\to\infty}\frac{1}{T}\int_0^T u'\,dt$$

$$\underbrace{\lim_{T\to\infty}\frac{1}{T}\int_0^T u\,dt}_{\overline{u}} - \overline{u} = \overline{u'}$$

$$\Rightarrow \qquad \underbrace{\overline{u} - \overline{u}}_{=0} = \overline{u'}$$

$$\Rightarrow \qquad \boxed{\overline{u'} = 0} \Leftarrow \text{mean of fluctuations is zero.}$$

Now, we will develop equations which govern the mean flow and try to develop some insight into how the fluctuations alter the mean flow equations.

Let's start with incompressible flow and look at the $x-$momentum:

$x-$momentum:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right]$$

Let's look at the "averaging" of this equation in time to develop an equation for the mean flow.

$$\Rightarrow \qquad \lim_{T\to\infty}\frac{1}{T}\int_0^T\left[x-mom\right]dt \text{ or } \underbrace{\overline{x-mom}}_{\substack{\text{time-average}\\ \text{of x-momentum}}}$$

The first term is:

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T\frac{\partial u}{\partial t}dt = \lim_{T\to\infty}\frac{1}{T}\int_0^T\frac{\partial(\overline{u}+u')}{\partial t}dt$$

But, $\frac{\partial \overline{u}}{\partial t}=0$ thus:

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T\frac{\partial u}{\partial t}dt = \lim_{T\to\infty}\frac{1}{T}\int_0^T\frac{\partial u'}{\partial t}dt = \overline{\frac{\partial u'}{\partial t}}$$

Just as $\overline{u'}=0$, we will <u>assume</u> $\overline{\frac{\partial u'}{\partial t}}=0$.

End result:

$$\overline{\frac{\partial u}{\partial t}}=0$$

Let's skip over to the pressure term and look at its average:

$$\frac{\partial p}{\partial x}=\frac{\partial \overline{p}}{\partial x}+\frac{\partial p'}{\partial x} \Rightarrow \overline{\frac{\partial p}{\partial x}=\frac{\partial \overline{p}}{\partial x}+\frac{\partial p'}{\partial x}}$$

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But, $\overline{p'}=0$ thus:

$$\overline{\frac{\partial p}{\partial x}}=\frac{\partial \overline{p}}{\partial x}$$

Similarly,

$$\overline{\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}}=\frac{\partial^2 \overline{u}}{\partial x^2}+\frac{\partial^2 \overline{u}}{\partial y^2}+\frac{\partial^2 \overline{u}}{\partial z^2}$$

Combining these into $x - mom$, we now have:

$$\underbrace{\overline{u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}}}_{\text{must still work this out}} = -\frac{1}{\rho}\frac{\partial \overline{p}}{\partial x} + \nu\left[\frac{\partial^2 \overline{u}}{\partial x^2} + \frac{\partial^2 \overline{u}}{\partial y^2} + \frac{\partial^2 \overline{u}}{\partial z^2}\right]$$

Let's work out the last term:

$$\overline{u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}} = \overline{u\frac{\partial u}{\partial x}} + \overline{v\frac{\partial u}{\partial y}} + \overline{w\frac{\partial u}{\partial z}}$$

That's the easy part. Now, consider $\overline{u\frac{\partial u}{\partial x}}$:

Question: What does $\overline{u\frac{\partial u}{\partial x}}$ equal in terms of $\overline{u}\ \&\ u'$ only?

a) $\overline{u}\frac{\partial \overline{u}}{\partial x}$

b) $\overline{u}\frac{\partial \overline{u'}}{\partial x} + \overline{u'}\frac{\partial \overline{u}}{\partial x}$

c) $\overline{u}\frac{\partial \overline{u}}{\partial x} + \overline{u'\frac{\partial u'}{\partial x}}$

d) none of the above