Sensitivity Analysis

Often in engineering analysis, we are not only interested in predicting the performance of a vehicle, product, etc, but we are also concerned with the sensitivity of the predicted performance to changes in the design and/or errors in the analysis. The quantification of the sensitivity to these sources of variability is called <u>sensitivity analysis</u>.

Let's make this more concrete using an example. Suppose we are interested in predicting the take off distance of an aircraft. From Anderson's, Intro. To Flight, an estimate for take-off distance is given by Eqn (6.104):

$$S_{LO} = \frac{1.44W^2}{g\rho_\infty S C_{L_{\max}} \mathrm{T}}$$

To make the example concrete, let's consider the jet aircraft CJ-1 described by Anderson. In that case, the conditions were:

$$\begin{aligned}
S &= 318\, ft^2 \\
\rho_\infty &= 0.002377\, slugs/ft^3 \;@\, sea\, level \\
g &= 32.2\, ft/s^2 \\
T &= 7300\ lbs \\
W &= 19815\, lbs
\end{aligned}$$

$$C_{L_{\max}} = 1.0$$

Thus, under these "nominal" conditions:

$$S_{LO} = \frac{1.44(19815)^2}{(32.2)(0.002377)(318)(1.0)(7300)}$$

$S_{LO} = 3182\, ft$ @ nominal conditions

Suppose that we were not confident of the $C_{L_{\max}}$ value and suspected that it might be ±0.1 from nominal.
That is,

$$0.9 \le C_{L_{\max}} \le 1.1$$

Also, let's suppose that the weight of the aircraft may need to be increased for a higher load. Specifically, let's consider a 10% weight increase:

1) Linear sensitivity analysis
2) Nonlinear sensitivity analysis (i.e. re-evaluation)

They both have their own advantage and disadvantages. The choice is often made based on the problem and the tools available. We'll look at both options.

<u>Linear Sensitivity Analysis</u>

Linear sensitivity based on Taylor series approximations. Suppose we were interested in the variation of S_{LO} with w & $C_{L_{\max}}$

Then:

$$S_{LO}(C_{L_{\max}} \Delta C_{L_{\max}}, W + \Delta W) \cong$$

$$S_{LO}(C_{L_{\max}}) + \frac{\partial S_{LO}}{\partial C_{L_{\max}}} \Delta C_{L_{\max}} + \frac{\partial S_{LO}}{\partial W} \Delta W$$

That is, the change in S_{LO} is:

$$\Delta S_{LO} \equiv \frac{\partial S_{LO}}{\partial C_{L_{\max}}} \Delta C_{L_{\max}} + \frac{\partial S_{LO}}{\partial W} \Delta W$$

The derivatives $\frac{\partial S_{LO}}{\partial C_{L_{\max}}} \& \frac{\partial S_{LO}}{\partial W}$ are the linear sensitivities of S_{LO} to changes in $C_{L_{\max}}$ & W_j, respectively.

Returning to the example:

$$\frac{\partial S_{LO}}{\partial C_{L_{\max}}} = \frac{-1.44W^2}{g\rho_\infty S C_{L_{\max}}{}^2 T} = -\frac{S_{LO}}{C_{L_{\max}}}$$

$$\frac{\partial S_{LO}}{\partial W} = \frac{2-1.44W}{g\rho_\infty S C_{L_{\max}} T} = 2\frac{S_{LO}}{W}$$

These can often be more information by looking at percent or fractional changes:

$$\frac{\Delta S_{LO}}{S_{LO}} \cong \frac{1}{S_{LO}} \frac{\partial S_{LO}}{\partial C_{L_{\max}}} \Delta C_{L_{\max}} + \frac{1}{S_{LO}} \frac{\partial S_{LO}}{\partial W} \Delta W$$

$$= \underbrace{\frac{C_{L_{\max}}}{S_{LO}} \frac{\partial S_{LO}}{\partial C_{L_{\max}}}} \frac{\Delta C_{L_{\max}}}{C_{L_{\max}}} + \underbrace{\frac{W}{S_{LO}} \frac{\partial S_{LO}}{\partial W} \frac{\Delta W}{W}}$$

Fractional sensitivities

For this problem:

$$\frac{C_{L_{\max}}}{S_{LO}}\frac{\partial S_{L_{\max}}}{\partial C_{L_{\max}}} = 1.0 \Rightarrow \boxed{\frac{\Delta S_{LO}}{S_{LO}} \approx -1\frac{\Delta C_{L_{\max}}}{C_{L_{\max}}}}$$

$$\frac{W}{S_{LO}}\frac{\partial S_{LO}}{\partial W} = 2.0 \Rightarrow \boxed{\frac{\Delta S_{LO}}{S_{LO}} \approx 2.0\frac{\Delta W}{W}}$$

Thus, a small fraction change in $C_{L_{\max}}$ will have an equal but opposite effect on the take-off distance.
The weight change will result in a charge of S_{LO} which is twice as large and in the same direction.
Thus, S_{LO} is more sensitive to W than $C_{L_{\max}}$ changes at least according to linear analysis.

Example

We were interested in $C_{L_{\max}}$ varying ± 0.1 which according to linear analysis would produce $\mu\, 0.1 S_{LO}$ variation in take-off distance:

$$\Rightarrow \quad \boxed{\begin{aligned} C_{L_{\max}} &= 0.9 \rightarrow \Delta S_{LO} \approx +0.1 S_{LO} = +318\,ft \\ C_{L_{\max}} &= 1.1 \rightarrow \Delta S_{LO} \approx +0.1 S_{LO} = -318\,ft \end{aligned}}$$

For a weight increase of 10% we find

$$\begin{aligned} W = (1.1)(19815\,lb) \rightarrow \Delta S_{LO} &\approx +2(0.1)S_{LO} \\ &\approx +636\,ft \end{aligned}$$

<u>Nonlinear Sensitivity Analysis</u>

For a nonlinear analysis, we simply re-evaluate the take off distance at the desired condition (including the perturbations). So, to assess the impact of the $C_{L_{\max}}$ variations we find:

$$S_{LO}(C_{L_{\max}} = 0.9) = \frac{1.44(19815)^2}{(32.2)(0.002377)(318)(0.9)(7300)}$$

$$S_{LO}(C_{L_{\max}} = 0.9) = 3535\,ft$$

$$\Rightarrow \boxed{\begin{array}{l} \Delta S_{LO}(\Delta C_{L_{\max}} = -0.1) = +353.6\,ft \\ \text{which agrees well with linear result} \end{array}}$$

Similarly,

$$S_{LO}\left(C_{L_{\max}} = 1.1\right) = 2892\,ft$$

$$\Rightarrow \boxed{\Delta S_{LO}\left(C_{L_{\max}} = +0.1\right) = -290\,ft}$$

Finally, a 10% W increase to 21796lb’s gives:

$$\boxed{\begin{array}{l} S_{LO}\left(W = 21796lb\right) = 3850\,ft \\ \Delta S_{LO}\left(\Delta W = +0.1W\right) = +668\,ft \end{array}}$$