Prandtl's Lifting Line Introduction

Assumptions:

- 3-D steady potential flow (inviscid & irrotational)
- Incompressible
- High aspect ratio wing
- Low sweep
- Small crossflow (along span)

  $\Rightarrow$ flow looks like 2-D flow locally with α adjusted

Outputs:

- Total lift and induced drag estimates
- Rolling moment
- Lift distribution along span
- Basic scaling of C_{D_i} with C_L & geometry

$$\boxed{C_{D_i} = \frac{C_L^2}{\pi A \mathrm{e}}} \Rightarrow \text{ dominated by } A \text{, where } A = b^2/S$$

$$\frac{D_i}{q_\infty S} = \frac{\left(L/q_\infty S \right)^2}{\pi \, b^2/S \, e}$$

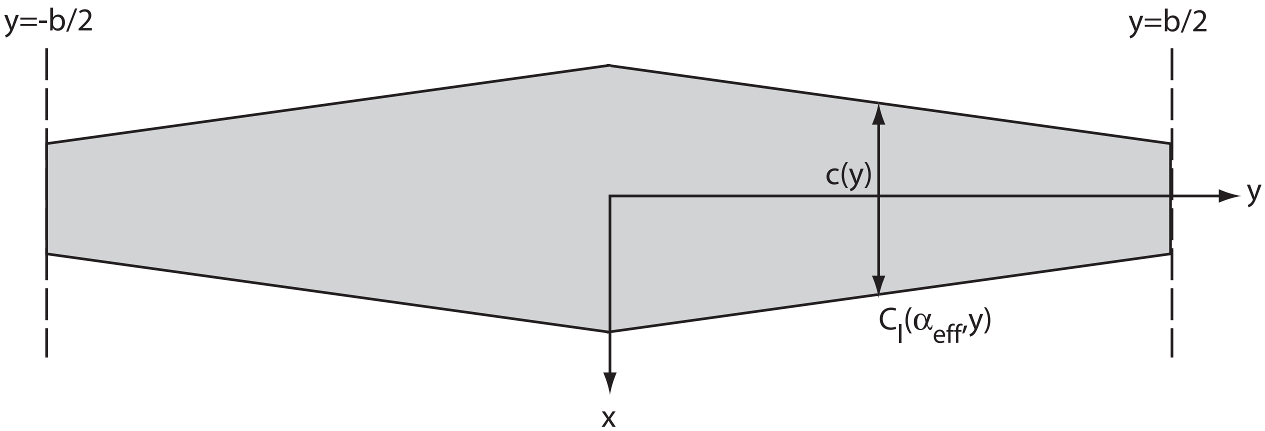
$$\Rightarrow D_i = \frac{L^2}{\pi b^2 q_\infty e} = \frac{1}{\pi q_\infty e}\left(\frac{L}{b}\right)^2$$

$$\boxed{D_i = \frac{1}{\pi q_\infty e}\left(\frac{L}{b}\right)^2}$$

In steady level flight where $L = W$, we have the following options for reducing D_i:

- Raise q_∞ (i.e. raise cruise velocity) $\Leftarrow$ Friction increases
- Decrease span loading, $\frac{W}{b}$ $\Leftarrow W \,\&\, b$ coupled due to structures
- Improve wing efficiency, e $\Leftarrow$Can be difficult

Geometry & Basic Definitions for Lifting Line



- Chord is variable, $c = c(y)$
- Angle of attack is variable, $\alpha = \alpha(y)$ and is a sum of two pieces

$$\underbrace{\alpha(y)}_{\substack{local\\ \alpha}} = \underbrace{\alpha_\infty}_{\substack{freestream\\ \alpha}} + \underbrace{\alpha_g(y)}_{\substack{geometric\\ twist}}$$

- Effective angle of attack is modified by downwash from trailing vortices

$$\alpha_{eff}(y) = \alpha(y) - \underbrace{\alpha_i(y)}_{\substack{induced\\ \alpha}}$$

Note: $\alpha_i > 0$ for downwash

- Local lift coefficient linear with α_{eff}

$$C_l = a_o(y)[\alpha_{eff}(y) - \alpha_{LO}(y)]$$

Fundamental Lifting Line Equation

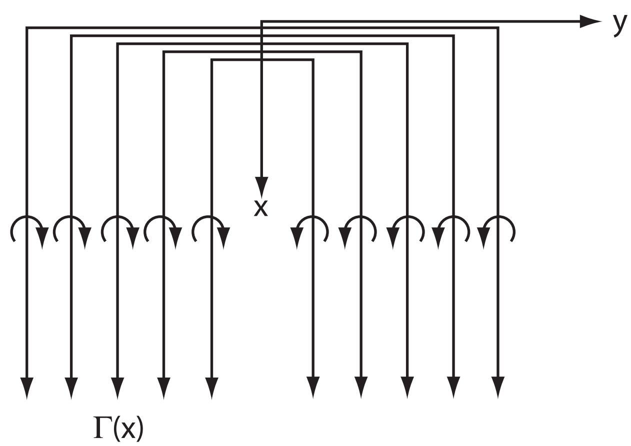
Basic model results by:

- Assuming Kutta-Joukowsky locally gives L':

$$L'(y) = \rho_\infty V_\infty \Gamma(y)$$

$$\Rightarrow C_l(y) = \frac{L'(y)}{q_\infty c(y)} = \frac{2\Gamma(y)}{V_\infty c(y)}$$

Distributing infinitely many horseshoe vortices along wing $c/4$-line to model induced flow:



$$C_l = a_o(y)\left[\alpha_{eff}(y) - \alpha_{LO}(y)\right] = \frac{2\Gamma(y)}{V_\infty c(y)}$$

$$= a_o(y)\left[\alpha(y) - \alpha_i(y) - \alpha_{LO}(y)\right] = \frac{2\Gamma(y)}{V_\infty c(y)}$$

$$\boxed{\alpha(y) = \frac{2\Gamma(y)}{a_o(y) V_\infty c(y)} + \alpha_{LO}(y) + \alpha_i(y)}$$

where $$\boxed{\alpha_i(y) = -\frac{w(y)}{V_\infty} = \frac{1}{4\pi V_\infty} \int_{-\frac{b}{2}}^{\frac{b}{2}} \frac{\frac{d\Gamma}{dy}(y_o) dy_o}{y - y_o}}$$

Note: only unknown is $\Gamma(y)$!

We use a Fourier series to solve this.

$$\Gamma = 2bV_\infty \sum_{n=1}^{N} A_n \sin n\theta \quad \text{where} \quad y = -\frac{b}{2}\cos\theta$$

Thus, the governing equation is:

$$\alpha(\theta) = \frac{4b}{a_o(\theta)c(\theta)} \sum_{n=1}^{N} A_n \sin n\theta + \alpha_{LO}(\theta) + \underbrace{\sum_{n=1}^{N} nA_n \frac{\sin n\theta}{\sin\theta}}_{\alpha_i(\theta)}$$