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16.07 Dynamics
Fall 2008
Version 2.0

Lecture L21 - 2D Rigid Body Dynamics

Introduction

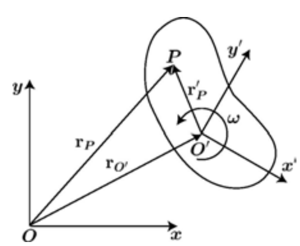
In lecture 11, we derived conservation laws for angular momentum of a **system** of particles, both about the center of mass, point G, and about a fixed (or at least non-accelerating) point O. We then extended this derivation to the motion of a rigid body in two-dimensional plane motion including both translation and rotation. We obtained statements about the conservation of angular momentum about both a fixed point and about the center of mass. Both are powerful statements. However, each has its own sperate requirements for application. In the case of motion about a fixed point, the point must have **zero acceleration**. Thus the instantaneous center of rotation, for example the point of contact of a cylinder rolling on a plane, cannot be used as the origin of our coordinates. For motion about the center of mass, no such restriction applies and we may obtain the statement of conservation of angular momentum about the center of mass even if this point is accelerating.

Kinematics of Two-Dimensional Rigid Body Motion

Even though a rigid body is composed of an infinite number of particles, the motion of these particles is constrained to be such that the body remains a rigid body during the motion. In particular, the only degrees of freedom of a 2D rigid body are *translation* and *rotation*.

Parallel Axes

Consider a 2D rigid body which is rotating with angular velocity ω about point O', and, simultaneously, point O' is moving relative to a fixed reference frame x and y with origin O.



In order to determine the motion of a point P in the body, we consider a second set of axes $x'y'$, always parallel to xy, with origin at O', and write,

$$\begin{aligned} \boldsymbol{r}_P &= \boldsymbol{r}_{O'} + \boldsymbol{r}'_P & (1)\\ \boldsymbol{v}_P &= \boldsymbol{v}_{O'} + (\boldsymbol{v}_P)_{O'} & (2)\\ \boldsymbol{a}_P &= \boldsymbol{a}_{O'} + (\boldsymbol{a}_P)_{O'} \ . & (3) \end{aligned}$$

Here, $\boldsymbol{r}_P$, $\boldsymbol{v}_P$ and $\boldsymbol{a}_P$ are the position, velocity and acceleration vectors of point P, as observed by O; $\boldsymbol{r}_{O'}$ is the position vector of point O'; and $\boldsymbol{r}'_P$, $(\boldsymbol{v}_P)_{O'}$ and $(\boldsymbol{a}_P)_{O'}$ are the position, velocity and acceleration vectors of point P, as observed by O'. Relative to point O', all the points in the body describe a circular orbit ($r'_P =$ constant), and hence we can easily calculate the velocity,

$$(v_P)_{O'} = r'_P\dot{\theta} = r\omega \ ,$$

or, in vector form,

$$(\boldsymbol{v}_P)_{O'} = \boldsymbol{\omega} \times \boldsymbol{r}'_P \ ,$$

where $\boldsymbol{\omega}$ is the angular velocity vector. The acceleration has a circumferential and a radial component,

$$((a_P)_{O'})_\theta = r'_P\ddot{\theta} = r'_P\dot{\omega}, \qquad ((a_P)_{O'})_r = -r'_P\dot{\theta}^2 = -r'_P\omega^2 \ .$$

Noting that $\boldsymbol{\omega}$ and $\dot{\boldsymbol{\omega}}$ are perpendicular to the plane of motion (i.e. $\boldsymbol{\omega}$ can change magnitude but not direction), we can write an expression for the acceleration vector as,

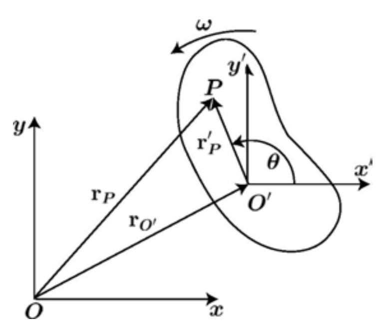
$$(\boldsymbol{a}_P)_{O'} = \dot{\boldsymbol{\omega}} \times \boldsymbol{r}'_P + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}'_P) \ .$$

Recall here that for any three vectors $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$, we have $\boldsymbol{A} \times (\boldsymbol{B} \times \boldsymbol{C}) = (\boldsymbol{A} \cdot \boldsymbol{C})\boldsymbol{B} - (\boldsymbol{A} \cdot \boldsymbol{B})\boldsymbol{C}$. Therefore $\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}'_P) = (\boldsymbol{\omega} \cdot \boldsymbol{r}'_P)\boldsymbol{\omega} - \omega^2\boldsymbol{r}'_P = -\omega^2\boldsymbol{r}'_P$. Finally, equations 2 and 3 become,

$$\begin{aligned} \boldsymbol{v}_P &= \boldsymbol{v}_{O'} + \boldsymbol{\omega} \times \boldsymbol{r}'_P & (4)\\ \boldsymbol{a}_P &= \boldsymbol{a}_{O'} + \dot{\boldsymbol{\omega}} \times \boldsymbol{r}'_P + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}'_P) \ . & (5) \end{aligned}$$

Body Axes

An alternative description can be obtained using body axes. Now, let $x'y'$ be a set of axes which are **rigidly attached** to the body and have the origin at point O'.



Then, the motion of an arbitrary point P can be expressed in terms of the general expressions for relative motion. Recall that,

$$\boldsymbol{r}_P = \boldsymbol{r}_{O'} + \boldsymbol{r}'_P \tag{6}$$

$$\boldsymbol{v}_P = \boldsymbol{v}_{O'} + (\boldsymbol{v}_P)_{O'} + \boldsymbol{\omega} \times \boldsymbol{r}'_P \tag{7}$$

$$\boldsymbol{a}_P = \boldsymbol{a}_{O'} + (\boldsymbol{a}_P)_{O'} + 2\boldsymbol{\omega} \times (\boldsymbol{v}_P)_{O'} + \dot{\boldsymbol{\omega}} \times \boldsymbol{r}'_P + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}'_P) \ . \tag{8}$$

Here, $\boldsymbol{r}_P$, $\boldsymbol{v}_P$ and $\boldsymbol{a}_P$ are the position, velocity and acceleration vectors of point P as observed by O; $\boldsymbol{r}_{O'}$ is the position vector of point O'; $\boldsymbol{r}'_P$, $(\boldsymbol{v}_P)_{O'}$ and $(\boldsymbol{a}_P)_{O'}$ are the position, velocity and acceleration vectors of point P as observed by O'; and $\boldsymbol{\Omega} = \boldsymbol{\omega}$ and $\dot{\boldsymbol{\Omega}} = \dot{\boldsymbol{\omega}}$ are the body angular velocity and acceleration.
Since we only consider 2D motions, the angular velocity vector, $\boldsymbol{\Omega}$, and the angular acceleration vector, $\dot{\boldsymbol{\Omega}}$, do not change direction. Furthermore, because the body is rigid, the relative velocity $(\vec{v}_P)_{O'}$ and acceleration $(\vec{a}_P)_{O'}$ of any point in the body, as observed by the body axes, is zero. Thus, equations 7 and 8 simplify to,

$$\boldsymbol{v}_P = \boldsymbol{v}_{O'} + \boldsymbol{\omega} \times \boldsymbol{r}'_P \tag{9}$$

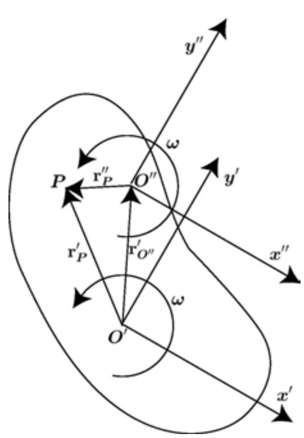
$$\boldsymbol{a}_P = \boldsymbol{a}_{O'} + \dot{\boldsymbol{\omega}} \times \boldsymbol{r}'_P + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}'_P) \ , \tag{10}$$

which are identical to equations 4 and 5, as expected. Note that their vector forms are equal. If at t=0, the frame x', y' (and eventually z') are instantaneously aligned with the frame x, y, the components of the vectors are qual. If not, then a coordinate transformation is required.

Invariance of $\boldsymbol{\omega}$ and $\boldsymbol{\alpha} = \dot{\boldsymbol{\omega}}$

The angular velocity, $\boldsymbol{\omega}$, and the angular acceleration, $\boldsymbol{\alpha} = \dot{\boldsymbol{\omega}}$, are invariant with respect to the choice of the reference point O'. In other words, this means that an observer using parallel axes situated *anywhere* in the rigid body will observe all the other points of the body turning around, in circular paths, with the same angular velocity and acceleration. Mathematically, this can be seen by considering an arbitrary point in the body O'' and writing,

$$\boldsymbol{r}'_P = \boldsymbol{r}'_{O''} + \boldsymbol{r}''_P \ .$$



Substituting into equations 6, 9 and 10, we obtain,

$$\begin{aligned}
\boldsymbol{r}_P &= \boldsymbol{r}_{O'} + \boldsymbol{r}'_{O''} + \boldsymbol{r}''_P & (11)\\
\boldsymbol{v}_P &= \boldsymbol{v}_{O'} + \boldsymbol{\omega} \times \boldsymbol{r}'_{O''} + \boldsymbol{\omega} \times \boldsymbol{r}''_P & (12)\\
\boldsymbol{a}_P &= \boldsymbol{a}_{O'} + \dot{\boldsymbol{\omega}} \times \boldsymbol{r}'_{O''} + \dot{\boldsymbol{\omega}} \times \boldsymbol{r}''_P + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}'_{O''}) + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}''_P)\ . & (13)
\end{aligned}$$

From equations 6, 9 and 10, replacing P with O'', we have that $\boldsymbol{r}_{O''} = \boldsymbol{r}_{O'} + \boldsymbol{r}'_{O''}$, $\boldsymbol{v}_{O''} = \boldsymbol{v}_{O'} + \boldsymbol{\omega} \times \boldsymbol{r}'_{O''}$, and $\boldsymbol{a}_{O''} = \boldsymbol{a}_{O'} + \dot{\boldsymbol{\omega}} \times \boldsymbol{r}'_{O''} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}'_{O''})$. Therefore, we can write,

$$\begin{aligned}
\boldsymbol{r}_P &= \boldsymbol{r}_{O''} + \boldsymbol{r}''_P & (14)\\
\boldsymbol{v}_P &= \boldsymbol{v}_{O''} + \boldsymbol{\omega} \times \boldsymbol{r}''_P & (15)\\
\boldsymbol{a}_P &= \boldsymbol{a}_{O''} + \dot{\boldsymbol{\omega}} \times \boldsymbol{r}''_P + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}''_P)\ . & (16)
\end{aligned}$$

These equations show that if the velocity and acceleration of point P are referred to point O'' rather than point O', then $\boldsymbol{r}'_P \neq \boldsymbol{r}''_P$, $\boldsymbol{v}_{O'} \neq \boldsymbol{v}_{O''}$, and $\boldsymbol{a}_{O'} \neq \boldsymbol{a}_{O''}$, although the angular velocity and acceleration vectors, $\boldsymbol{\omega}$ and $\boldsymbol{\alpha}$, remain unchanged.

Instantaneous Center of Rotation

We have established that the motion of a solid body can be described by giving the position, velocity and acceleration of *any* point in the body, plus the angular velocity and acceleration of the body. It is clear that if we could find a point, C, in the body for which the instantaneous velocity is zero, then the velocity of the body at that particular instant would consist only of a rotation of the body about that point (no translation). If we know the angular velocity of the body, $\boldsymbol{\omega}$, and the velocity of, say, point O', then we could determine the location of a point, C, where the velocity is zero. From equation 9, we have,

$$0 = \boldsymbol{v}_{O'} + \boldsymbol{\omega} \times \boldsymbol{r}'_C.$$

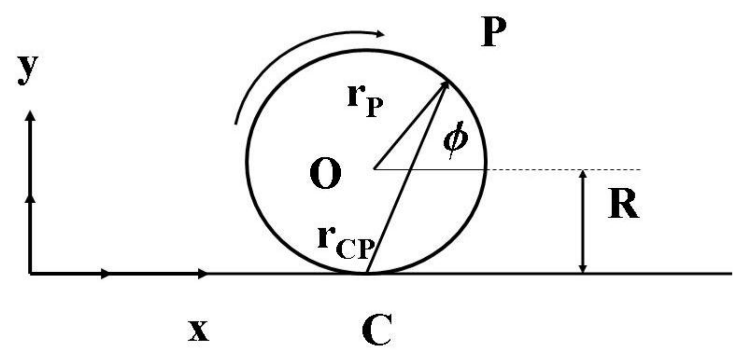
Point C is called the *instantaneous center of rotation*. Multiplying through by $\boldsymbol{\omega}$, we have $-\boldsymbol{\omega} \times \boldsymbol{v}_{O'} = \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}'_C)$, and, re-arranging terms, we obtain,

$$\boldsymbol{r}'_C = \frac{1}{\omega^2}(\boldsymbol{\omega} \times \boldsymbol{v}_{O'})\ ,$$

which shows that $\boldsymbol{r}'_C$ and $\boldsymbol{v}_{O'}$ are perpendicular, as we would expect if there is only rotation about C. Alternatively, if we know the velocity at two points of the body, P and P', then the location of point C can be determined geometrically as the intersection of the lines which go through points P and P' and are perpendicular to $\boldsymbol{v}_P$ and $\boldsymbol{v}_{P'}$. From the above expression, we see that when the angular velocity, ω, is very small, the center of rotation is very far away, and, in particular, when it is zero (i.e. a pure translation), the center of rotation is at infinity. Although the center of rotation is a useful concept, if the pointO' is accelerating, it cannot be used as the origin in the application of the principle of conservation of angular momentum.

Example ***Rolling Cylinder***

Consider a cylinder rolling on a flat surface, without sliding, with angular velocity $\boldsymbol{\omega}$ and angular acceleration $\boldsymbol{\alpha}$. We want to determine the velocity and acceleration of point P on the cylinder. In order to illustrate the various procedures described, we will consider three different approaches.



Direct Method :

Here, we find an expression for the position of P as a function of time. Then, the velocity and acceleration are obtained by simple differentiation. Since there is no sliding, we have,

$$\boldsymbol{v}_{O'} = -\omega R\,\boldsymbol{i}, \qquad \boldsymbol{a}_{O'} = -\alpha R\,\boldsymbol{i},$$

and,

$$\boldsymbol{r}_P = \boldsymbol{r}_{O'} + R\cos\phi\,\boldsymbol{i} + R\sin\phi\,\boldsymbol{j}\ .$$

Therefore,

$$\begin{aligned}
\boldsymbol{v}_P = \dot{\boldsymbol{r}}_P &= \boldsymbol{v}_{O'} - R\dot{\phi}\sin\phi\,\boldsymbol{i} + R\dot{\phi}\cos\phi\,\boldsymbol{j} \\
&= -\omega R(1+\sin\phi)\,\boldsymbol{i} + \omega R\cos\phi\,\boldsymbol{j} \\
\boldsymbol{a}_P = \ddot{\boldsymbol{r}}_P &= \boldsymbol{a}_{O'} - R(\ddot{\phi}\sin\phi + \dot{\phi}\cos\phi)\,\boldsymbol{i} + R(\ddot{\phi}\cos\phi - \dot{\phi}\sin\phi)\,\boldsymbol{j} \\
&= [-\alpha R(1+\sin\phi) - \omega^2 R\cos\phi]\,\boldsymbol{i} + [\alpha R\cos\phi - \omega^2 R\sin\phi]\,\boldsymbol{j}
\end{aligned}$$

Relative motion with respect to C :

Here, we use expressions 4 and 5, or 9 and 10, with O' replaced by C.

$$\boldsymbol{v}_P = \boldsymbol{\omega} \times \boldsymbol{r}'_{CP} \tag{17}$$

$$\boldsymbol{a}_P = \boldsymbol{a}_C + \dot{\boldsymbol{\omega}} \times \boldsymbol{r}'_{CP} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}'_{CP})\ . \tag{18}$$

In the above expressions, we have already used the fact that $\boldsymbol{v}_C = \boldsymbol{0}$. Now,

$$\boldsymbol{r}'_{CP} = R\cos\phi\,\boldsymbol{i} + R(1+\sin\phi)\,\boldsymbol{j}\,, \qquad \boldsymbol{\omega} = \omega\boldsymbol{k}\ ,$$

and,

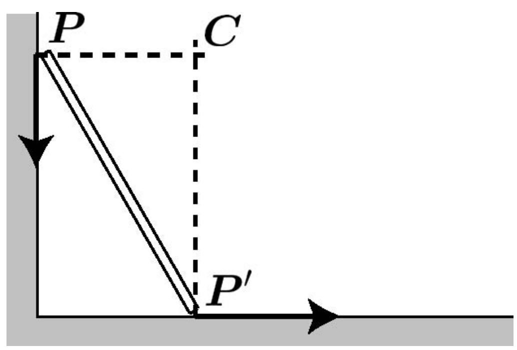
$$\boldsymbol{v}_P = -\omega R(1+\sin\phi)\,\boldsymbol{i} + \omega R\cos\phi\,\boldsymbol{j}\ .$$

The calculation of $\boldsymbol{a}_P$, in this case, requires knowing $\boldsymbol{a}_C$. In the no sliding case, $\boldsymbol{a}_C$ can be shown to be equal to $R\omega^2\,\boldsymbol{j}$, i.e., it only has a vertical component. With this, after some algebra, we obtain,

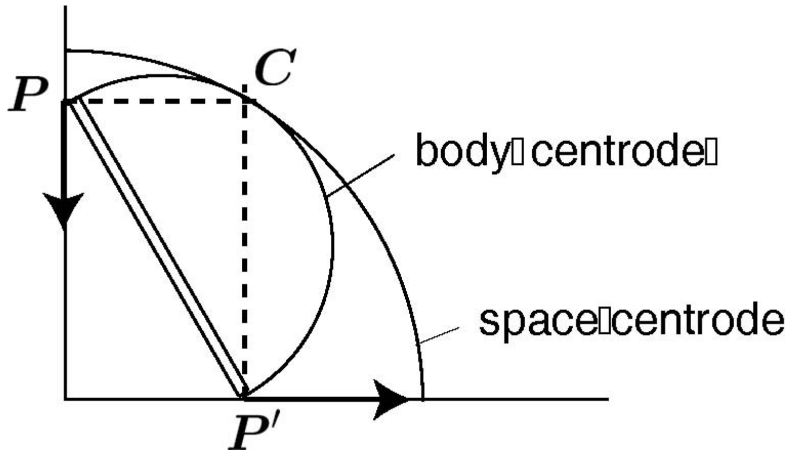
$$\boldsymbol{a}_P = [-\alpha R(1+\sin\phi) - \omega^2 R\cos\phi]\,\boldsymbol{i} + [\alpha R\cos\phi - \omega^2 R\sin\phi]\,\boldsymbol{j}\ .$$

Example ***Sliding bar***

Consider a bar leaning against the wall and slipping downward. It is clear that while the bar is in contact with the wall and the floor, the velocity at point P will be in the vertical direction, whereas the velocity at point P' will be in the horizontal direction. Therefore, drawing the perpendicular lines to $\boldsymbol{v}_P$ and $\boldsymbol{v}_{P'}$ through points P and P', we can determine the instantaneous center of rotation C.



It should be noted that, for a general motion, the location of the center of rotation will change in time. The path described by the instantaneous center of rotation is called the *space centrode*, and the locus of the positions of the instantaneous centers on the body is called the *body centrode*. At a given instant, the space centrode and the body centrode curves are tangent. The tangency point is precisely the instantaneous center of rotation, C. Therefore at this instance, the point C is common to both curves. It is not difficult to show that, for the above example, the space and body centrodes are circular arcs, assuming that the points P and P' remain in contact with the walls at all times.



From this example, it should be clear that although we think about the instantaneous center of rotation as a point attached to the body, it need not be a material point. In fact, it can be a point "outside" the body. It is also possible to consider the instantaneous center of acceleration as the point at which the instantaneous acceleration is zero.

Review: Conservation of Angular Momentum for 2D Rigid Body

In Lecture 11, we showed that the equations describing the general motion of a rigid body follow from the conservation laws for systems of particles. Since the general motion of a **2D rigid body** can be determined by three parameters (e.g. x and y coordinates of position, and a rotation angle θ), we will need to supply three equations. Conservation of linear momentum yields one vector equation, or two scalar equations. The additional condition is conservation of angular momentum. In Lecture 11, We saw that there are several ways to express conservation of angular momentum. In principle, they are all equivalent, but, depending on the problem situation, the use of a particular form may greatly simplify the problem. The best choices for the origin of coordinates are: 1) the center of mass G; 2) a fixed point O.

Conservation of Angular Momentum about the Center of Mass

When considering a 2D rigid body, the velocity of any point relative to G consists of a pure rotation and, therefore, the conservation law for angular momentum about the center of mass, G is

$$\boldsymbol{H}_G = \sum_{i=1}^{n} (\boldsymbol{r}'_i \times m_i(\boldsymbol{\omega} \times \boldsymbol{r}'_i)) = \boldsymbol{\omega} \sum_{i=1}^{n} m_i r_i'^2 = \boldsymbol{\omega} \int_m r'^2 dm \tag{19}$$

For a continuous body, the sum over the mass points is replaced by an integral. $\int_m r'^2 dm$ is defined as the mass moment of inertia I_G about the center of mass.

$$I_G \boldsymbol{\alpha} = \boldsymbol{M}_G, \tag{20}$$

where $I_G = \int_m r'^2 dm$, $\boldsymbol{\alpha} = \dot{\boldsymbol{\omega}}$ and M_G is the total moment about G due to external forces and external moments. Although equation (7) is a vector equation, $\boldsymbol{\alpha}$ and $\boldsymbol{M}_G$ are always perpendicular to the plane of motion, and, therefore, equation (7) only yields one scalar equation. The moment of inertia, I_G, can be interpreted as a measure of the body's resistance to changing its angular velocity as a result of applied external moments. The moment of inertia, I_G, is a scalar quantity. It is a property of the solid which indicates the way in which the mass of the solid is distributed relative to the center of mass. For example, if most of the mass is far away from the center of mass, r'_i will be large, resulting in a large moment of inertia. The dimensions of the moment of inertia are $[M][L^2]$.

Conservation of Angular Momentum about a fixed point O

If the fixed point O is chosen as the origin, a similar result is obtained. Since for a 2D rigid body the velocity of any point in the coordinate system fixed at the point O is

$$\boldsymbol{v}_i = \boldsymbol{\omega} \times \boldsymbol{r}_i \ ,$$

conservation of angular momentum gives

$$I_O \boldsymbol{\alpha} = \boldsymbol{M}_O, \tag{21}$$

where $I_O = \int_m r^2 dm$, $\boldsymbol{\alpha} = \dot{\boldsymbol{\omega}}$ and M_O is the total applied moment due to external forces and moments (torques). Also it is important to point out that both the angular velocity $\boldsymbol{\omega}$ and the angular acceleration $\boldsymbol{\alpha}$ are the same for any point on a rigid body: $\boldsymbol{\omega}_G = \boldsymbol{\omega}_O$, $\boldsymbol{\alpha}_G = \boldsymbol{\alpha}_O$.

Most textbooks on dynamics have tables of moments of inertia for various common shapes: cylinders, bars, plates. See Meriam and Kraige, *Engineering Mechanics,* DYNAMICS (Appendix B) for more examples.

Radius of Gyration

It is common to report the moment of inertia of a rigid body in terms of the radius of gyration, k. This is defined as

$$k = \sqrt{\frac{I}{m}} \ ,$$

and can be interpreted as the root-mean-square of the mass element distances from the axis of rotation. Since the moment of inertia depends upon the choice of axis, the radius of gyration also depends upon the choice of axis. Thus we write

$$k_G = \sqrt{\frac{I_G}{m}} \ ,$$

for the radius of gyration about the center of mass, and

$$k_O = \sqrt{\frac{I_O}{m}} \ ,$$

for the radius of gyration about the fixed point O.

Parallel Axis Theorem

We will often need to find the moment of inertia with respect to a point other than the center of mass. For instance, the moment of inertia with respect to a given point, O, is defined as

$$I_O = \int_m r^2 \, dm \ .$$

Assuming that O is a fixed point, $\boldsymbol{H}_O = I_O \boldsymbol{\omega}$. If we know I_G, then the moment of inertia with respect to point O, can be computed easily using the parallel axis theorem. Given the relations $r^2 = \boldsymbol{r} \cdot \boldsymbol{r}$ and $\boldsymbol{r} = \boldsymbol{r}_G + \boldsymbol{r}'$., we can then write,

$$I_O = \int_m r^2 \, dm = \int_m (r_G^2 + 2\boldsymbol{r}_G \cdot \boldsymbol{r}' + r'^2) \, dm = r_G^2 \int_m dm + 2\boldsymbol{r}_G \cdot \int_m \boldsymbol{r}' \, dm + \int_m r'^2 \, dm = m r_G^2 + I_G \ ,$$

since $\int_m \boldsymbol{r}' \, dm = \boldsymbol{0}$.

From this expression, it also follows that the moment of inertia with respect to an arbitrary point is **minimum** when the point coincides with G. Hence, the **minimum** value for the moment of inertia is I_G.

Summary: Governing Equations

Now that we have reviewed the equation governing conservation of angular momentum for a 2D rigid body in planar motion about both the center of mass and about a fixed point O , we can restate the governing equations for this three degree of freedom system.

The conservation of linear momentum yields the vector equation,

$$m\boldsymbol{a}_G = \boldsymbol{F} \ , \tag{22}$$

where m is the body mass, $\boldsymbol{a}_G$ is the acceleration of the center of mass, and $\boldsymbol{F}$ is the sum of the external forces acting on the body.

Conservation of angular momentum about the center of mass requires

$$\dot{\boldsymbol{H}}_G = \boldsymbol{M}_G = \dot{\boldsymbol{\omega}} I_G = \boldsymbol{\alpha} I_G \tag{23}$$

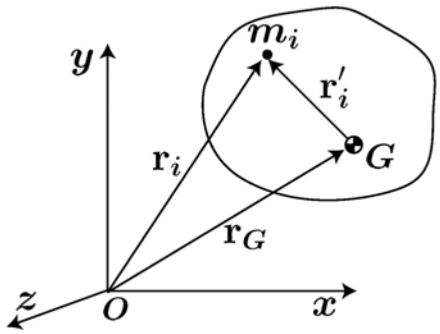
where I_G is the moment of inertia about the center of mass; $\boldsymbol{\omega}$ is the angular velocity, whose vector direction is perpendicular to the x, y coordinate system; and $\boldsymbol{a}$ is the angular acceleration.

A body fixed at a point O is a single degree of freedom system. Therefore, only one equation is required, conservation of angular momentum about the point O.

$$\dot{\boldsymbol{H}}_O = \boldsymbol{M}_O = \dot{\boldsymbol{\omega}} I_O = \boldsymbol{\alpha} I_O \tag{24}$$

Kinetic Energy for a 2D Rigid Body

We start by recalling the kinetic energy expression for a system of particles derived in lecture L11,



$$T = \sum_{i=1}^{n} \frac{1}{2} m_i (\boldsymbol{v}_G + \dot{\boldsymbol{r}}'_i) \cdot (\boldsymbol{v}_G + \dot{\boldsymbol{r}}'_i) = \frac{1}{2} m v_G^2 + \frac{1}{2} \boldsymbol{\omega} \int_m r'^2 dm = \frac{1}{2} m v_G^2 + \frac{1}{2} \boldsymbol{\omega}^2 I_G. \quad (25)$$

where n is the total number of particles, m_i denotes the mass of particle i, and $\boldsymbol{r}'_i$ is the position vector of particle i with respect to the center of mass, G. Also, $m = \sum_{i=1}^{n} m_i$ is the total mass of the system, and $\boldsymbol{v}_G$ is the velocity of the center of mass. The above expression states that the kinetic energy of a system of particles equals the kinetic energy of a particle of mass m moving with the velocity of the center of mass, plus the kinetic energy due to the motion of the particles relative to the center of mass, G.

When the body is rotating about a fixed point O, we can write $I_O = I_G + m r_G^2$ and

$$T = \frac{1}{2} m v_G^2 + \frac{1}{2} (I_O - m r_G^2) \omega^2 = \frac{1}{2} I_O \omega^2 ,$$

since $v_G = \omega r_G$.

The above expression is also applicable in the more general case when there is no fixed point in the motion, provided that O is replaced by the instantaneous center of rotation. Thus, in general,

$$T = \frac{1}{2} I_C \omega^2 .$$

We shall see that, when the instantaneous center of rotation is known, the use of the above expression does simplify the algebra considerably.

Work

External Forces

Since the body is rigid and the internal forces act in equal and opposite directions, only the external forces applied to the rigid body are capable of doing any work. Thus, the total work done on the body will be

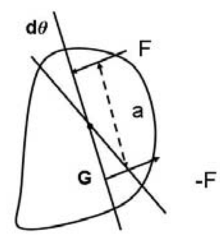
$$\sum_{i=1}^{n} (W_i)_{1-2} = \sum_{i=1}^{n} \int_{(\boldsymbol{r}_i)_1}^{(\boldsymbol{r}_i)_2} \boldsymbol{F}_i \cdot d\boldsymbol{r} \quad ,$$

where $\boldsymbol{F}_i$ is the sum of all the **external** forces acting on particle i.

Work done by couples

If the sum of the external forces acting on the rigid body is zero, it is still possible to have non-zero work. Consider, for instance, a moment $M = Fa$ acting on a rigid body. If the body undergoes a pure translation, it is clear that all the points in the body experience the same displacement, and, hence, the total work done by a couple is zero. On the other hand, if the body experiences a rotation $d\theta$, then the work done by the couple is

$$dW = F\,\frac{a}{2}d\theta + F\,\frac{a}{2}d\theta = F\,ad\theta = Md\theta \ .$$



If M is constant, the work is simply $W_{1-2} = M(\theta_2 - \theta_1)$. In other words, the couples do work which results in the kinetic energy of rotation.

Conservative Forces

When the forces can be derived from a potential energy function, V, we say the forces are conservative. In such cases, we have that $\boldsymbol{F} = -\boldsymbol{\nabla} V$, and the work and energy relation in equation **??** takes a particularly simple form. Recall that a necessary, but not sufficient, condition for a force to be conservative is that it must be a function of position only, i.e. $\boldsymbol{F}(\boldsymbol{r})$ and $V(\boldsymbol{r})$. Common examples of conservative forces are gravity (a constant force independent of the height), gravitational attraction between two bodies (a force inversely proportional to the squared distance between the bodies), and the force of a perfectly elastic spring.

The work done by a conservative force between position $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ is

$$W_{1-2} = \int_{\boldsymbol{r}_1}^{\boldsymbol{r}_2} \boldsymbol{F}\cdot d\boldsymbol{r} = [-V]_{\boldsymbol{r}_1}^{\boldsymbol{r}_2} = V(\boldsymbol{r}_1) - V(\boldsymbol{r}_2) = V_1 - V_2 \quad .$$

Thus, if we call W_{1-2}^{NC} the work done by all the external forces which are *non conservative*, we can write the general expression,

$$T_1 + V_1 + W_{1-2}^{NC} = T_2 + V_2 \quad .$$

Of course, if all the forces that do work are conservative, we obtain conservation of total energy, which can be expressed as,

$$T + V = \text{constant} \; .$$

Gravity Potential for a Rigid Body

In this case, the potential V_i associated with particle i is simply $V_i = m_i g z_i$, where z_i is the height of particle i above some reference height. The force acting on particle i will then be $\boldsymbol{F}_i = -\boldsymbol{\nabla} V_i$. The work done on the whole body will be

$$\sum_{i=1}^{n} \int_{\boldsymbol{r}_i^1}^{\boldsymbol{r}_i^2} \boldsymbol{F}_i \cdot d\boldsymbol{r}_i = \sum_{i=1}^{n} ((V_i)_1 - (V_i)_2) = \sum_{i=1}^{n} m_i g((z_i)_1 - (z_i)_2 = V_1 - V_2 \quad ,$$

where the gravity potential for the rigid body is simply,

$$V = \sum_{i=1}^{n} m_i g z_i = m g z_G \; ,$$

where z_G is the z coordinate of the center of mass. It's obvious but worth noting that because the gravitational potential is taken about the center of mass, the inertia plays no role in determining the gravitational potential.

ADDITIONAL READING

J.L. Meriam and L.G. Kraige, *Engineering Mechanics, DYNAMICS*, 5th Edition

5/1, 5/2, 5/3, 5/4 (review) , 5/5, 5/6 (review), 6/6, 6/7

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Fall 2009