

**CHRISTINE
BREINER:**

Welcome back to recitation. In this video, I'd like us to work on the following problem: So given the region R shown below, which I'll mention more of in a moment, compute the integral double integral over R of the quantity $4x^2 - y^4$ $dx dy$, and I want you to do it by changing variables to $u = 2x - y$ and $v = 2x + y$.

And the region, defined down here, is just these three line segments and the intersection of the region bounded by these three line segments. So one of them is a portion of the y -axis, one of them is a portion of the line defined by $2x - y = 0$, and one of them is a portion of the line defined by $2x + y = 2$. So why don't you do this, make sure you use this change of variables, and then I'll come back, and I'll draw the accompanying region in the u, v plane, and then we'll see how I set it up.

OK, welcome back. Well, the first thing I'm going to do is I'm going to draw the region in the u, v plane that is determined by this region in the x, y plane and the change of variables. Now, as was mentioned in the lecture, all of the changes are linear, and so I'm going to be taking lines to lines. So what I'm going to do is determine at each endpoint here where that endpoint goes under the transformation, and then I will connect the dots.

So what do I get? Well, this first one is $(0, 0)$ for x and y , and so that corresponds to x and y are 0 here, so u is 0, and x and y are 0 here, so it would be a 0, so that's the point $(0, 0)$ on the u, v plane.

Now, I want to mention something here is that based on how these lines were defined, this is telling you that all along this line $2x - y = 0$, that means u is equal to 0 all along that segment. And if you'll notice also, $2x + y = 2$ here, so v is equal to 2 all along this segment. So we should expect whatever u is, v 's going to be 2 here, and whatever v is here, u is going to be 0. OK, so I'm going to have something with $u, 0$ initially, and then $v, 2$, I can already expect, it's going to come up here and move over that way. But let's just check.

So this is the point $(0, 2)$, 0 for x and 2 for y . Let's see what it is in the u, v variables. So u in that case is negative 2 and v in that case is 2, and so I'm going to go left 2 and up 2. This might not be drawn to scale, but this is negative 2, and this is up 2 here, and so that's where this point goes in the transformation. This point is $(-2, 2)$. You can actually check what it is. But actually, as I mentioned, you already know what has to happen, because this segment is connected right here, and then we said all along this line, v is 2, and all along this line, u is 0,

and so that actually carves out that rectangle. So if you were worried and you weren't able to do it that way, what you could actually do is say, well, where does the point $(1/2, 1)$ go? If I plug in $1/2$ for x and 1 for y , I notice that I get u equals 0 and v equals 2 , and that's this point.

And so now, I'm looking at this region. Now, these two things are not drawn to scale, because what I want to point out is they're both triangles. It's easy to find the area based on how they're sitting in two-dimensional space. So what we see here is if I wanted to find the area of this triangle, let's make this the base. The base is 2 and the height is $1/2$, so $1/2$ of base times height is-- that area is equal to $1/2$. So the area of R is $1/2$, and what's the area of this one? The area this one is-- well, I've got base 2 and height 2 , and so it's $1/2$ of 2 times 2 , and so that gives me area as 2 .

So here the area is $1/2$, here the area is 2 , and so notice that I've multiplied by 4 to get from here to here. So I can anticipate that I should have-- based on the principle you saw in lecture, I should have something like $du dv$ is going to be equal to 4 times $dx dy$. That is what I expect to get. Now let's see if when we do the Jacobian, we get the same thing. Let me double check, right? This is area 2 , this is area $1/2$, so I have $1/2$ times 4 is going to give me 2 . Yeah, that's what I should get.

Oops! That sticks out a little. So now let's just check our Jacobian and see if that is indeed what we get. So I'm going to look at u sub x , u sub y , v sub x , v sub y , right? So u sub x is 2 , u sub y is negative 1 , so I get 2 , negative 1 . v sub x is 2 , v sub y is 1 , so I get 2 , 1 . 2 times 1 is 2 , minus negative 2 gives me, indeed, 4 . So I do get, in fact, $du dv$ is equal to $4 dx dy$. So I know that I'm going to have to-- because I'm changing variables, though, from $dx dy$ to $du dv$, I'm going to divide by 4 , obviously, when I do the substitution. The substitution will be replacing just $dx dy$. So I just mention that, but notice: Again, we get what we expect to get. We got a 4 based on the picture and we got a 4 based on the Jacobian.

And so now, we need to finish up the process. We need to figure out how to write this in terms of u and v and then figure out our bounds in terms of u and v , and then we're done. So let me mention, one thing you should notice is that u times v is equal to exactly-- this is going to be a difference of two squares, and it is precisely $4x^2 - y^2$. If you need to multiply it out to check, you can check it, but that's indeed what it is, which is why this particular substitution is quite nice, because that means this function I'm supposed to be integrating is just u^4 . So I'm almost done. So let me write in the pieces I know, and then we'll fill in the last two spots, or four spots, which will be all the bounds on the integral.

So I'm going to write it here, give myself some space to write the bounds. So I'm replacing the $4x^2 - y^2$ by a $u \cdot v$, so I'm going to get u times v , and then the function, that is raised to the fourth in the initial problem, so I raise that to the fourth. Then $dx dy$, as I mentioned, will be replaced by a $du dv$ divided by 4. And so I can just put this over 4 and write $du dv$. I should be careful which order I want to do. It doesn't really matter. I can do either one. $du dv$, OK?

And so now I want to know what u goes from and to and then what v has to go from and to. So if I come over here, you'll see it didn't matter, because I could have picked either direction to go. But if I'm going to go with u , I'm coming from whatever this function is over to here, right? And this value here is easy. That's $u = 0$. So the top bound for u is 0, and the bottom bound for u , this is $v = -u$, so the bottom bound for u is when u is equal to negative v . So I'm running from-- because I put the du on the inside, my first one is running from minus v to-- what did I say-- 0, and then the v -values from there go between 0 and 2. You can see this easily from the picture. So they go between 0 and 2.

And I am not going to finish it off from here, because it would require not too much work. It's actually quite simple, because it's just a polynomial and u and a polynomial and v , but there are a lot of powers and there will be big powers of 2. So suffice it to say, at this point, we can evaluate this integral and it's quite easy to evaluate.

And the main point I think we should make is how did it make it simpler? I mean, the initial problem, if we come over here, this is annoying to find an anti-derivative, but not impossible. But the really annoying part is I would have to take this region and split it into a bunch of pieces, or at least two pieces, to evaluate the integral in a reasonable way or my y -values would go from this line here up to this line here. It could be complicated.

So what we've really done is we've simplified the region. That's the easiest thing. I mean, we also simplified the function, but we've really simplified the region we're integrating over. And so we only have-- one is a function and one is a constant, and that's quite nice to have on the inside. So I think that's where I'll stop.