

Problems: Polar Coordinates and the Jacobian

1. Let $r = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1} \frac{y}{x}$. Directly calculate the Jacobian $\frac{\partial(r, \theta)}{\partial(x, y)} = \frac{1}{r}$.

Answer: Because we are familiar with the change of variables from rectangular to polar coordinates and we know that $\frac{\partial(r, \theta)}{\partial(x, y)} \cdot \frac{\partial(x, y)}{\partial(r, \theta)} = 1$, this result should not come as a surprise.

$$\begin{aligned} \frac{\partial(r, \theta)}{\partial(x, y)} &= \begin{vmatrix} r_x & r_y \\ \theta_x & \theta_y \end{vmatrix} \\ &= \begin{vmatrix} \frac{2x}{2\sqrt{x^2+y^2}} & \frac{2y}{2\sqrt{x^2+y^2}} \\ \frac{-y/x^2}{1+(y/x)^2} & \frac{1/x}{1+(y/x)^2} \end{vmatrix} \\ &= \begin{vmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} \\ -\frac{y}{x^2+y^2} & \frac{x}{x^2+y^2} \end{vmatrix} \\ &= \begin{vmatrix} \frac{x}{r} & \frac{y}{r} \\ -\frac{y}{r^2} & \frac{x}{r^2} \end{vmatrix} \\ &= \frac{x^2 + y^2}{r^3} = \frac{1}{r}. \end{aligned}$$

2. For the change of variables $x = u$, $y = \sqrt{r^2 - u^2}$, write $dx dy$ in terms of u and r .

Answer: We know $dx dy = \left| \frac{\partial(x, y)}{\partial(u, r)} \right| du dr$.

$$\begin{aligned} \frac{\partial(x, y)}{\partial(u, r)} &= \begin{vmatrix} x_u & x_r \\ y_u & y_r \end{vmatrix} \\ &= \begin{vmatrix} 1 & 0 \\ \frac{-u}{\sqrt{r^2-u^2}} & \frac{r}{\sqrt{r^2-u^2}} \end{vmatrix} \\ &= \frac{r}{\sqrt{r^2 - u^2}} \end{aligned}$$

Hence $dx dy = \frac{r}{\sqrt{r^2 - u^2}} du dr$.

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