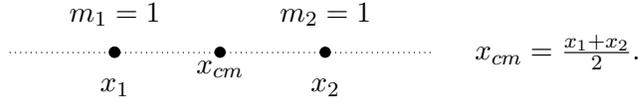


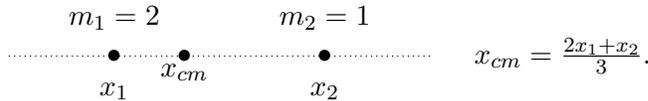
## Mass and average value

### Center of Mass

**Example 1:** For two equal masses, the center of mass is at the midpoint between them.



**Example 2:** For unequal masses the center of mass is a weighted average of their positions.



In general,  $x_{cm}$  = weighted average of position =  $\frac{\sum m_i x_i}{\sum m_i}$ .

For a continuous density,  $\delta(x)$ , on the segment  $[a, b]$  (units of density are mass/unit length) the sums become integrals. We will skip running through the logic of this since we are about to show it for two dimensions.

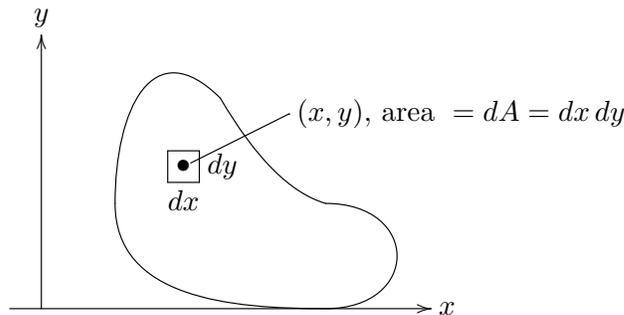
$$M = \int_a^b \delta(x) dx, \quad x_{cm} = \frac{1}{M} \int_a^b x \delta(x) dx. \quad \frac{\delta(x)}{a \qquad \qquad \qquad b}$$

In 2 dimensions we label the center of mass as  $(x_{cm}, y_{cm})$  and we have the following formulas

$$M = \iint_R \delta(x, y) dA, \quad x_{cm} = \frac{1}{M} \iint_R x \delta(x, y) dA, \quad y_{cm} = \frac{1}{M} \iint_R y \delta(x, y) dA.$$

These formulas are easy to justify using our usual method for building integrals.

In this case, we divide our region into little pieces and we sum up the contributions of each piece using an integral. To keep the figure below uncluttered we only show one piece and we don't bother to label it as the  $i^{\text{th}}$ . In the end we will go directly to the integral, by thinking if it as a sum.



The little piece shown has mass  $\delta(x, y) dA$  and the total mass is just the sum the pieces. That is, it's just the integral

$$M = \iint_R \delta dA.$$

Likewise the  $x$  and  $y$  coordinates of the center of mass are just the weighted average of the  $x$  and  $y$  coordinates of each of the pieces. So, we get the formulas given above.

**Example:** Suppose the unit square has density  $\delta = xy$ ; Find its mass and center of mass.

**Answer:**

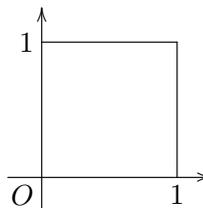
$$M = \iint_R \delta \, dA = \int_0^1 \int_0^1 xy \, dx \, dy = \frac{1}{4}.$$

$$x_{cm} = \frac{1}{M} \iint_R x \delta \, dA = \frac{1}{M} \int_0^1 \int_0^1 x^2 y \, dy \, dx.$$

This is easily computed as  $x_{cm} = \frac{2}{3}$ .

By symmetry of the region and the density  $\delta$ , we also have

$$y_{cm} = \frac{2}{3}.$$



### Average Value

We can think of center of mass as the average position of the mass. That is, it's the average of position with respect to mass. We can also take averages of functions with respect to other things. For instance, the *average value* of  $f(x, y)$  with respect to area on a region  $R$  is

$$\frac{1}{\text{area } R} \iint_R f(x, y) \, dA.$$

In general, if we simply say the average value of a function, it means average value with respect to area (or later, when we do triple integrals it can mean with respect to volume).

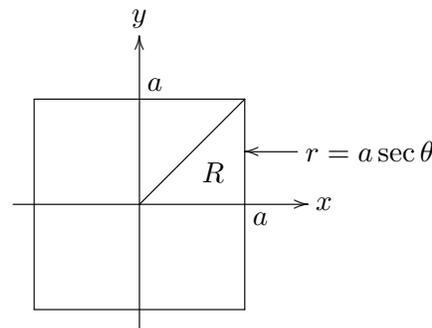
**Example:** What's the average distance of a point in a unit square from the center?

**Answer:** We center the square on the origin. Notice each side has length  $2a$  and the area of the square is  $4a^2$ . So,

$$\text{average distance} = \frac{1}{4a^2} \iint_{\text{square}} \sqrt{x^2 + y^2} \, dx \, dy.$$

By symmetry the integral is 8 times the integral of the triangular region  $R$  shown. We actually compute the integral in polar coordinates.

$$\begin{aligned} \text{average} &= \frac{8}{4a^2} \iint_R r \, r \, dr \, d\theta \\ &= \frac{2}{a^2} \int_0^{\pi/4} \int_0^{a \sec \theta} r^2 \, dr \, d\theta \\ &= \frac{2}{a^2} \int_0^{\pi/4} \frac{a^3 \sec^3 \theta}{3} \, d\theta \\ &= \frac{a}{3} (\sqrt{2} + \ln(\sqrt{2} + 1)). \end{aligned}$$



This last integral was computed using integration by parts as

$$\int \sec^3 \theta \, d\theta = \frac{\int \sec \theta \, d\theta + \sec \theta \tan \theta}{2} = \frac{\ln(\sec \theta + \tan \theta) + \sec \theta \tan \theta}{2}.$$

Note: the center of mass is the average value of  $x$  and  $y$  with respect to mass.

The *geometric center* has coordinates given by the average value of  $x$  and  $y$  with respect to area, i.e., the center of mass when  $\delta = 1$ .

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