

JOEL LEWIS: Hi. Welcome back to recitation.

I've got a nice exercise here on Stokes' Theorem for you. Now this problem is a little bit more sophisticated than a lot of problems we've been doing in recitation. So it requires a little bit more thought, and it also involves more mathematical sophistication. So we're doing a clever kind of proof here that I like. So hopefully you'll like this one. It's a little bit in a different style than some of the ones we've done. So I think I need to talk about it a little bit before we get started.

So let's let F be the field $[x, y, z]$. So this is our radial field that we've seen a lot in recitation. So what I'd like you to do is prove that this field is not the curl of any field G . All right. So I'd like you to show that there's no field G such that F is equal to the curl of G .

Now, rather than just saying that to you and letting you run off, I have a suggestion for an interesting way you could go about this. And this interesting way is going to use Stokes' Theorem. So what I'd like you to do is a proof by contradiction.

OK, so what you're going to do is you're going to assume that F is a curl. OK? So you're going to assume that there is some G such that F is curl G . And then you're going to use that to get a ridiculous conclusion. So you're going to start with that premise, and you're going to end up with a contradiction.

So these two arrows colliding into each other is a symbol that mathematicians use for a contradiction. So you're going to start from this premise, and you're going to reach a contradiction. And what that's going to show is that your premise couldn't be right. Yeah? Because if you start from a true premise, well then all your conclusions should be true as well. So if you reach a false conclusion, then you must have had a false premise.

So what you're going to do is you're going to take a sphere of radius b centered at the origin. And a curve C on the sphere. You know, a simple, closed curve. So assuming that F is this curl of G , what I'd like you to do is use Stokes' Theorem to interpret the line integral of $G \cdot dr$ over C in two different ways. OK? And interpreting this line integral in two different ways, you're going to reach a contradiction, and that will show that F really isn't a curl.

So that's what I'd like you to do. So why don't you pause the video, go ahead and see if you can work that out, come back, and we'll talk about it together.

I hope you enjoyed working on this problem. Let's get started on it.

So as I was saying before we started, what we're going to do is we're looking for a proof by contradiction. So as the problem says, we're going to start with a sphere. And I'm going to take this curve C -- some simply connected closed curve that's going to go around the back of the sphere, and it's going to be oriented one way or the other-- and it's going to divide this sphere into two pieces.

So there's the one cap on one side of it, S_1 . And then there's-- whatever the other piece on the other side of it is, S_2 . OK. And so what we're going to do is we're going to think about, what is this line integral? OK. So this is our curve C here on the sphere.

So the integral over C of $G \cdot dr$. So this is what the problem suggests we think about. So this is a line integral of a field dot dr over the boundary of a surface. Well, actually, it's the boundary of two surfaces. Right? C is the boundary of S_1 , and C -- if we orient it the other way-- is the boundary of S_2 , when we orient them both outwards. OK, so what is this?

So Stokes' Theorem tells us something about this line integral. So let's first think about this as the top cap-- that cap S_1 -- with boundary C oriented so that they agree with each other. So the normal is outwards on the sphere, and C is proceeding in the direction that I've drawn the arrow here.

Well, in that circumstance, we have that the integral around C of $G \cdot dr$, by Stokes' Theorem, is equal to the surface integral over S_1 of $\text{curl of } G \cdot n$ with respect to surface area. Right? So this is just Stokes' Theorem. Stokes' Theorem says the line integral of G around the boundary curve is equal to the surface integral of the curl of G over the region, provided all of our orientations are correct. OK.

Well, we know though what curl of G is, because by assumption, F is equal to curl of G . OK, so this is equal to the surface integral over S_1 of $F \cdot n \, dS$. So in the first step, we use Stokes' Theorem. In the second step, we use our assumption that curl G is equal to F . Well, now what?

But we know what F is. Right? F is this radial field $[x, y, z]$. So F and n are pointing in the same direction. They're parallel to each other. n is a unit vector, so this is just the length of F . This $F \cdot n$ is just the length of F . And since we're on a sphere of radius b , this is just b . OK, so the

integrand is just b .

So this is the integral over S_1 of $b \, dS$, which is b times the area of S_1 . OK. One thing I'd like you to notice is that in particular, this is a positive number. b is positive and the area is positive. OK. So that's our first interpretation.

So we took our field G that we suppose exists, and we integrate it around this curve C , and we apply Stokes' Theorem, and then the fact that we know what F is means that we know what $F \cdot n$ is, and so that makes our surface integral very easy to compute. And it turns out to be b times the area of S , which, I just happened to notice, is a positive number. OK.

Well, now we can do the same trick on the other half of the sphere. Right? So we just did the top cap here. We did S_1 . So now we have the bottom cap, or whatever. All the rest of the sphere, S_2 . OK.

So we can also get that the integral over C of $G \cdot dr$, we can interpret it in terms of Stokes' Theorem. But notice then that C -- we still want to use the same normal. We like outwards pointing normals. So we're going to have to orient C the other way in order to make Stokes' Theorem make sense. So let's walk over here where we have some empty board space.

So we want to orient C the other way. So in other words, we're going to take the negative of this line integral. So it's minus $G \cdot dr$. And if we apply Stokes' Theorem to this line integral-- so this is the same line integral, but with the opposite orientation on C and so with the opposite sign-- by Stokes' Theorem, this is equal to the integral over S_2 of the curl of $G \cdot n \, dS$.

And so the way I've set this up, this is still my outward-pointing normal. OK. But again, we can use our assumption and we can get curl of G is equal to F , because we're assuming that G has this property. So this is equal to the integral over S_2 of $F \cdot n$ with respect to surface area. And again, n is the outward-pointing normal, and F is parallel to it.

So this dot product is just the length of F . The outward pointing unit normal. So this is just the length of F , which is b . So this is equal to b times the area of S_2 , which is also positive. So what have we just shown?

Well, we started from the assumption that there exists a G such that F is the curl of G . And starting from that assumption-- let's look-- we showed that the line integral around C of $G \cdot dr$ is equal to some positive number. And we also showed, over here, that the negative of the line integral of G around C is equal to some positive number. Well, this is clearly absurd. That

can't be true.

So starting from our assumption that F was the curl of G -- that there is a G such the F is the curl of G -- we reached an absurd conclusion. We reached a conclusion that the same number is both positive and negative. But that can't happen. So that means our premise had to be false. OK.

So this is a contradiction. So our assumption is false. And our assumption was-- that we used to get this whole thing started-- that F was the curl of some G . All right. So what have we shown?

So we used a nice argument here with Stokes' Theorem in order to show that certain fields aren't the curl of other fields. So Stokes' Theorem limits the kind of fields that can be curls of other fields. Now perhaps, you may have thought of other theorems that you can use that also limit what sorts of fields can be curls.

And so there are other ways to reach this true conclusion that our field F -- whose components are x , y , and z -- is not the curl of any field. This isn't the only way to reach that conclusion. But this is a nice way that shows that Stokes' Theorem puts some limitations on what fields can behave like if they're going to be curls.

I'll stop there.