

JOEL LEWIS: Hi. Welcome back to recitation. In lecture, you've been learning about using the method of Lagrange multipliers to optimize functions of several variables given a constraint. So here's a problem that you can practice this method on.

So I've got a function f of x, y, z equals x squared plus x plus $2y$ squared plus $3z$ squared. And what I'd like you to do is find the maximum and minimum values that this function takes as the point (x, y, z) moves around the unit sphere x squared plus y squared plus z squared equals 1 . So to optimize this function given the constraint x squared plus y squared plus z squared equals 1 .

So why don't you pause the video, take some time to work that out, come back, and we can work it out together.

Hopefully you had some luck working on this problem. Let's have a go at it.

So remember that the method of Lagrange multipliers-- in order to apply it-- what it says is that when you have a function being optimized on some constraint condition, what you do to find the points where the function could be maximum or minimum is that first you look for points where the gradient of your objective function is parallel to the gradient of your constraint function.

So what that means is you take the partial derivatives f_x, f_y, f_z , and you say f_x has to be equal to λ times g_x , f_y has to be equal to λ times g_y , and f_z has to be equal to λ times g_z , for some λ . And then you solve that system together with the constraint equation. And so the points that are the solutions of that system of equations, those points are your points that you have to check for whether they're the maximum or the minimum.

And also, sometimes you have some boundary to your region and you have to check that as well. So in this case, the sphere doesn't have boundary. Right? So we don't have any boundary conditions to check. So we're going to have a really straightforward problem to solve where we just have to look at the partial derivatives. So let's write down that system of equations that we have to solve.

So the partial derivative of f with respect to x is going to be $2x$ plus 1 . So we have to solve the system $2x$ plus 1 equals-- and the partial derivative of our constraint with respect to x is $2x$, so

$2x + 1$ has to equal λ times $2x$. That's what we get from the x -partial derivatives.

How about from the y -partial derivatives? The y -partial derivative of f is going to be $4y$. So that has to be equal to λ and the y -partial derivative of the constraint equation which is $2y$.

And the z -partial derivative of f is $6z$. So $6z$ has to be equal to λ times, well, the z -partial derivative of the constraint function, which is $2z$. And we have the last equation $x^2 + y^2 + z^2 = 1$.

So we get four equations in our variables x , y , and z , plus this new parameter λ that we introduced. And we want to solve these to find the points x , y , and z at which these equations are all satisfied. And then, once we get those points, we have to test them to see whether they are the maximum or the minimum or neither.

So OK. So we have this system of equations. Now, this is a little bit complicated. It's not a system of linear equations. So we need to think about ways that we can solve it. And one thing that I think we can do here, is if you look at the second and third equations, you see that in the second equation, everything has a factor of y in it. So either y is equal to 0, or we can divide by it.

So from the second equation, we have that either y is equal to 0, or we can divide by y , in which case we get $\lambda = 2$. Similarly, from the third equation, we have that either z is equal to 0, or we can divide by z and we get $\lambda = 3$. So from the third equation, we have $z = 0$ or $\lambda = 3$.

So now we have a bunch of possibilities, right? So either we have $y = z = 0$, or we have $y = 0$ and $\lambda = 3$, or we have $\lambda = 2$ and $z = 0$. Or, well, the other possibility would be $\lambda = 2$ and $\lambda = 3$, but that can't happen. So we have three possibilities. Three different ways that this could be satisfied. So let's go over here and write down what those possibilities are.

So case one, or maybe I'll call it case a. So the case a is when $y = z = 0$. So when $y = z = 0$ -- OK, we need to find x still. So let's look back at our equations.

And when $y = z = 0$, well, we can solve our constraint equation for x . When $y = z = 0$, we have that $x^2 = 1$. So there are two possibilities. The point

(1, 0, 0), and the point minus 1, 0, 0. So this gives us, in this case, we have x equals 1 or x equals minus 1. So that gives us the points (1, 0, 0) and minus 1, 0, 0 that we're going to have to check at the end. All right.

So the second case is we could have y equal to 0 and λ equal to 3. So in this case, let's go back to our equations again. So from λ equals 3, we have in our first equation that $2x$ plus 1 equals $6x$. So 1 equals $4x$ or x equals $1/4$. So this implies over here that x equals $1/4$.

And now, we still need to find z . So if we go back to our constraint equation here, we have that x is a quarter and y is 0. So that means $1/16$ plus z squared equals 1. So z has to be the square root of $15/16$, plus or minus. And z is equal to plus or minus-- so that we can also write that as the square root of 15 over 4.

So this also gives us two points to check. The points are $1/4$, 0, the square root of 15 over 4. And $1/4$, 0, minus square root of 15 over 4.

And finally, we have our third case. So our third case is when λ is equal to 2 and z is equal to 0. So again, let's go back over to our equation. So when λ equals 2 in the first equation, we have $2x$ plus 1 equals $4x$. So $2x$ equals 1 or x is $1/2$. So this gives us x equals a half.

And now if you take z equals 0 and x equals $1/2$, we can take that down to our constraint equation. And we get a quarter plus y squared equals 1, so y is a square root of $3/4$. So y equals plus or minus square root of 3 over 2.

And this gives us two points. $1/2$, square root of 3 over 2, 0. And $1/2$, minus square root of 3 over 2, 0.

Those were our three cases. We've solved each of them. We've solved each of them all the way down to finding the points that they lead to. Now remember, we said already that there's no boundary to this region. It's just the sphere. It has no edges.

So these are the only points we have to check. We have to check these six points. What do we have to check them for? Well, we have to look at the value of f at each of these six points. And we want to figure out where f is maximized and where f is minimized, and these six points are the only points where that could happen, where f could be maximized or minimized.

So we just have to evaluate our objective function f at these six points and find the largest

value and the smallest value. So let's do that. So our objective function, remember, it's all the way back over here. It's this function $x^2 + x + 2y^2 + 3z^2$. OK. So let's look at the value of that function at these point.

So $x^2 + x + 2y^2 + 3z^2$ at the point $(1, 0, 0)$, that's just equal to 2. So I'm going to write the function values just off to the side of the points here. So this gives me the value 2. And I'm going to circle them. So the point $(1, 0, 0)$ gives me the value 2.

The point $(-1, 0, 0)$ -- so that's x^2 is 1, plus x is minus 1-- so that's $1 - 1 = 0$ -- and then the y and z terms are both 0. So at the point $(-1, 0, 0)$, the function value is 0. I'm going to circle that. Oh boy.

OK, so at these points-- at the point $(\frac{1}{4}, 0, \frac{\sqrt{15}}{4})$, and $(\frac{1}{4}, 0, -\frac{\sqrt{15}}{4})$ -- I'm going to cheat and look at what I wrote down already. So you could do the arithmetic yourself, but I think it's not that hard to work out that in both of these cases, the function value that you get out is $\frac{25}{8}$. I'm not going to do the arithmetic right now. But you can double-check that for yourself.

And at these last two points-- the points $(\frac{1}{2}, \frac{\sqrt{3}}{2}, 0)$, and $(\frac{1}{2}, -\frac{\sqrt{3}}{2}, 0)$ -- the function has the same value at both of those points. That value is $\frac{9}{4}$. Yeah, so $\frac{25}{8}$ was the value at both of these points, and $\frac{9}{4}$ is the value of both of these points.

So now, to find the maximum value of the function and the minimum value of the function, we just look at the values that we got and say, which of these is biggest and which of these is smallest? And in our case, it's easy to see that 0 is the minimum. You know, all the other values are positive, so 0 is the minimum. So our minimum value of f is 0 at the point $(-1, 0, 0)$.

And if you just compare the values 2 and $\frac{25}{8}$ and $\frac{9}{4}$, $\frac{25}{8}$ is the largest. This is bigger than 3, whereas both of those are less than 3, for example. This is one easy way to see that. So the max of f is $\frac{25}{8}$, and that's achieved at the points $(\frac{1}{4}, 0, \pm\frac{\sqrt{15}}{4})$.

So there you have it. The method of Lagrange multipliers. We just followed exactly the strategy that we have.

So you start out and you have an objective function and a constraint function. And so what do you do? You write down their partial derivatives and you come up with this system of

equations. So this system of equations that you get by setting, you know, f_x equal to λg_x , f_y equal to λg_y , f_z equals λg_z , and your constraint equation g equals some constant.

So then the one part of this procedure that isn't just a recipe is that you need to solve this system of equations, but sometimes that can be hard. So in this case, there were a couple of observations that we could make from the second and third equations that made it relatively straightforward to do. And that gave us some cases.

And then in each of those cases, we were able to completely solve for the points x , y , and z . Now we also could solve for the associated values of λ , but λ isn't important to us. It doesn't affect f . We can forget about it as soon as we found it, once we found x , y , and z . So we were able to solve.

In this case, we got six points of interest. And then you just look at the value of your objective function at those points. So that was what I wrote down in these circles. So you look at the value of the objective function. And to find the maximum value of the function, you just look at which of those is largest.

Now sometimes-- not in this problem, but in other problems, you'll also have to check-- if the region has a boundary, you'll also have to check for possible maxima and minima on the boundary of the region. But a sphere doesn't have any edges, so it doesn't have any boundary. So we don't have to worry about that.

So that's how we apply the method of Lagrange multipliers to this problem. And how you can apply it to other problems as well. I'll end there.