

## V2.2-3 Gradient Fields and Exact Differentials

### 2. Finding the potential function.

In example 2 in the previous reading we saw that

$$\mathbf{F} = \frac{x \mathbf{i} + y \mathbf{j}}{x^2 + y^2} = \nabla \ln \sqrt{x^2 + y^2} = \nabla \ln r.$$

This raises the question of how we found the function  $\frac{1}{2} \ln(x^2 + y^2)$ . More generally, if we know that  $\mathbf{F} = \nabla f$  — for example if  $\text{curl } \mathbf{F} = 0$  in the whole  $xy$ -plane — how do we find the function  $f(x, y)$ ? There are two methods; some students prefer one, some the other.

**Method 1.** Suppose  $\mathbf{F} = \nabla f$ . By the Fundamental Theorem for Line Integrals,

$$(5) \quad \int_{(x_0, y_0)}^{(x, y)} \mathbf{F} \cdot d\mathbf{r} = f(x, y) - f(x_0, y_0).$$

Read from left to right, (5) gives us an easy way of finding the line integral in terms of  $f(x, y)$ . But read right to left, it gives us a way of finding  $f(x, y)$  by using the line integral:

$$(5') \quad f(x, y) = \int_{(x_0, y_0)}^{(x, y)} \mathbf{F} \cdot d\mathbf{r} + c.$$

(Here  $c$  is an arbitrary constant of integration; as (5') shows,  $c = f(x_0, y_0)$ .)

**Remark.** It is common to refer to  $f(x, y)$  as the (mathematical) **potential function**. The potential function used in physics is  $-f(x, y)$ . The negative sign is used by physicists so that the potential difference will represent work done *against* the field  $\mathbf{F}$ , rather than work done *by* the field, as the convention is in mathematics.

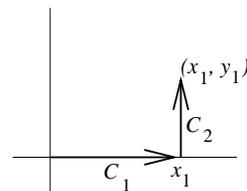
**Example 3.** Let  $\mathbf{F} = (x + y^2) \mathbf{i} + (2xy + 3y^2) \mathbf{j}$ . Verify that  $\mathbf{F}$  satisfies the Criterion (2), and use method 1 above to find the potential function  $f(x, y)$ .

**Solution.** We verify (2) immediately:  $\frac{\partial(y^2)}{\partial y} = 2y = \frac{\partial(2xy)}{\partial x}$ .

We use (5'). The point  $(x_0, y_0)$  can be any convenient starting point;  $(0, 0)$  is the usual choice, if the integrand is defined there. (We will subscript the variables to avoid confusion with the variables of integration, but you don't have to.) By (5'),

$$(6) \quad f(x_1, y_1) = \int_{(0,0)}^{(x_1, y_1)} (x + y^2) dx + (2xy + 3y^2) dy.$$

Since the integral is path-independent, we can choose any path we like. The usual choice is the one on the right, as it simplifies the computations. (Most of what follows you can do mentally, with a little practice.)



On  $C_1$ , we have  $y = 0$ ,  $dy = 0$ , so the integral on  $C_1$  becomes  $\int_0^{x_1} x dx = \frac{1}{2} x_1^2$ .

On  $C_2$ , we have  $x = x_1$ ,  $dx = 0$ , so the integral is  $\int_0^{y_1} (2x_1y + 3y^2) dy = x_1 y_1^2 + y_1^3$ .

Adding the integrals on  $C_1$  and  $C_2$  to get the integral along the entire path, and dropping the subscripts, we get by (6) and (5')

$$f(x, y) = \frac{1}{2}x^2 + xy^2 + y^3 + c .$$

(The constant of integration is added by (5'), since the choice of starting point was arbitrary. You should always confirm the answer by checking that  $\nabla f = \mathbf{F}$ .)  $\square$

**Method 2.** Once again suppose  $\mathbf{F} = \nabla f$ , that is  $M \mathbf{i} + N \mathbf{j} = f_x \mathbf{i} + f_y \mathbf{j}$  . It follows that

$$(7) \quad f_x = M \quad \text{and} \quad f_y = N .$$

These are two equations involving partial derivatives, which we can solve simultaneously by integration. We illustrate using the previous example:  $\mathbf{F} = (x + y^2, 2xy + 3y^2)$ .

**Solution by Method 2.** Using the first equation in (7),

$$(8) \quad \begin{aligned} \frac{\partial f}{\partial x} &= x + y^2. && \text{Hold } y \text{ fixed, integrate with respect to } x: \\ f &= \frac{1}{2}x^2 + y^2x + g(y). && \text{where } g(y) \text{ is an arbitrary function of } y. \end{aligned}$$

To find  $g(y)$ , we calculate  $\frac{\partial f}{\partial y}$  two ways:

$$\begin{aligned} \frac{\partial f}{\partial y} &= 2yx + g'(y) && \text{by (8), while} \\ \frac{\partial f}{\partial y} &= 2xy + 3y^2 && \text{from (7), second equation.} \end{aligned}$$

Comparing these two expressions, we see that  $g'(y) = 3y^2$ , so  $g(y) = y^3 + c$ . Putting it all together, using (8), we get  $f(x, y) = \frac{1}{2}x^2 + y^2x + y^3 + c$ , as before.  $\square$

In the first method, the answer is written down immediately as a line integral; the rest of the work is in evaluating the integral, which goes quickly, since on a horizontal or vertical path either  $dx = 0$  or  $dy = 0$ .

In the second method, the answer is obtained by an algorithm involving several steps which should be carried out in the right order.

The first method has the advantage of reminding you each time how  $f(x, y)$  is defined and what it means, facts of theoretical and practical importance. The second method has the advantage of requiring no knowledge of line integrals, which makes it popular with students; on the other hand, when done in three dimensions, the bookkeeping gets more complicated, whereas in the first method it does not; overall, the first method is faster, provided you are confident enough to do some of the work mentally.

### 3. Exact differentials.

The formal expressions  $M(x, y) dx + N(x, y) dy$  which have appeared as the integrands in our line integrals are called **differentials**. In some applications, most notably thermodynamics, one usually works directly with the differential  $M dx + N dy$  and its line integral  $\int M dx + N dy$ , without considering or using the associated vector field  $\mathbf{F} = M \mathbf{i} + N \mathbf{j}$ . Therefore it is important to have the results about gradient fields in this section translated into the language of differentials. We do this now.

If  $f(x, y)$  is a differentiable function, its **total differential**  $df$  (or simply *differential*) is by definition the expression

$$(9) \quad df = f_x dx + f_y dy .$$

For example, if  $f(x, y) = x^2 y^3$ , then  $d(x^2 y^3) = 2xy^3 dx + 3x^2 y^2 dy$ .

We call the differential  $M dx + N dy$  **exact**, in a region  $D$  where  $M$  and  $N$  are defined, if it is the total differential of some function  $f(x, y)$  in this region, i.e., if in  $D$ ,

$$(10) \quad M = f_x \quad \text{and} \quad N = f_y, \quad \text{for some } f(x, y).$$

From this we see that the relation between differentials and vector fields is

$$\begin{aligned} M dx + N dy \text{ is exact} &\Leftrightarrow M \mathbf{i} + N \mathbf{j} \text{ is a gradient field} \\ M dx + N dy = df &\Leftrightarrow M \mathbf{i} + N \mathbf{j} = \nabla f . \end{aligned}$$

In this language, the criterion (we use the same equation number as in the section where they were first presented)

$$(2) \quad \mathbf{F} = \nabla f \text{ for some } f(x, y) \Rightarrow M_y = N_x .$$

and its partial converse:

Let  $\mathbf{F} = M \mathbf{i} + N \mathbf{j}$  be continuously differentiable for all  $x, y$ .

$$(4) \quad M_y = N_x \text{ for all } x, y \Rightarrow \mathbf{F} = \nabla f \text{ for some differentiable } f \text{ and all } x, y.$$

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**Exactness Criterion.** Assume  $M$  and  $N$  are continuously differentiable in a region  $D$  of the plane. Then in this region,

$$(11) \quad M dx + N dy \text{ exact} \Rightarrow M_y = N_x ;$$

$$(12) \quad \text{if } D \text{ is the whole } xy\text{-plane, } M_y = N_x \Rightarrow M dx + N dy \text{ exact.}$$

If the exactness criterion shows that  $M dx + N dy$  is exact, then the function  $f(x, y)$  may be found by either of the two methods previously described.

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