

Product rule for vector derivatives

1. If $\mathbf{r}_1(t)$ and $\mathbf{r}_2(t)$ are two parametric curves show the product rule for derivatives holds for the cross product.

Answer: As with the dot product, this will follow from the usual product rule in single variable calculus. We want to show

$$\frac{d(\mathbf{r}_1 \times \mathbf{r}_2)}{dt} = \mathbf{r}'_1 \times \mathbf{r}_2 + \mathbf{r}_1 \times \mathbf{r}'_2.$$

Let $\mathbf{r}_1 = \langle x_1, y_1, z_1 \rangle$ and $\mathbf{r}_2 = \langle x_2, y_2, z_2 \rangle$. We have,

$$\mathbf{r}_1 \times \mathbf{r}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} = \langle y_1 z_2 - z_1 y_2, z_1 x_2 - x_1 z_2, x_1 y_2 - y_1 x_2 \rangle.$$

Taking derivatives using the product rule from single variable calculus, we get a lot of terms, which we can group to prove the vector formula.

$$\begin{aligned} \frac{d(\mathbf{r}_1 \times \mathbf{r}_2)}{dt} &= \langle y'_1 z_2 + y_1 z'_2 - z'_1 y_2 - z_1 y'_2, z'_1 x_2 + z_1 x'_2 - x'_1 z_2 - x_1 z'_2, x'_1 y_2 + x_1 y'_2 - y'_1 x_2 - y_1 x'_2 \rangle \\ &= \langle (y'_1 z_2 - z'_1 y_2) + (y_1 z'_2 - z_1 y'_2), (z'_1 x_2 - x'_1 z_2) + (z_1 x'_2 - x_1 z'_2), (x'_1 y_2 - y'_1 x_2) + (x_1 y'_2 - y_1 x'_2) \rangle \\ &= \langle x'_1, y'_1, z'_1 \rangle \times \langle x_2, y_2, z_2 \rangle + \langle x_1, y_1, z_1 \rangle \times \langle x'_2, y'_2, z'_2 \rangle \\ &= \mathbf{r}'_1 \times \mathbf{r}_2 + \mathbf{r}_1 \times \mathbf{r}'_2. \quad \blacksquare \end{aligned}$$

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