

JOEL LEWIS: Hi. Welcome back to recitation. In lecture, you've been learning about triple integration. And I have a problem here for you on computing a volume of a region using a triple integral. So let's look at this.

So I have a volume and I'm describing it to you; it's the volume inside the paraboloid  $z$  equals  $x$  squared plus  $y$  squared and bounded by the plane  $z$  equals  $2y$ . So I've drawn a little picture here for you. So this is the paraboloid here. And we're just taking a plane cut of it. And so this is going to slice off some chunk of that paraboloid, and what I want to know is what the volume is of that piece that gets cut off by that plane there. So below the plane and above the paraboloid.

So, why don't you pause the video, take some time, work out this problem, come back, and we can work on it together.

I hope you had some luck with this problem. I think it's a bit of a tricky one, so let's start to work through it together.

So sometimes you have a problem with a triple integral. And you need to set up your bounds of integration. And sometimes you can look at it and it's just clear what they are. If you're integrating over a cube, life is really easy.

But in this case, this region that we want to integrate over is kind of more complicated to understand. Right? So it's easy to see-- well, relatively easy to see-- what the bounds on  $z$  are.

So let me draw a couple of two-dimensional pictures. So I'm going to draw the  $yz$ -plane cross section here. So in the  $yz$ -plane cross section, this paraboloid just becomes a parabola.

So that becomes the parabola  $z$  equals  $y$  squared, which is a plane section of the paraboloid  $z$  equals  $x$  squared plus  $y$  squared. And this plane  $z$  equals  $2y$  becomes the line  $z$  equals  $2y$ . And this little sliver is a

plane section of the region in question. So we see that  $z$  is going from the paraboloid to the plane. And over here, we see that  $z$  is going from the paraboloid to the plane.

But what we really need to understand then is what the relationship between  $x$  and  $y$  is. So what is the shadow of this region? How are  $x$  and  $y$  related to each other? How can we bound  $x$  in terms of  $y$  or  $y$  in terms of  $x$ ? Or should we use cylindrical coordinates or what?

And so in order to that what we need to do is we need to figure out-- when you project this region down, when you flatten it along  $z$ , so you're disregarding  $z$  now, and then you're just looking at its shadow, its footprint, in the  $xy$  plane-- you want to figure out, what is that region? What does it look like?

So somehow we'll project down and there will be some region  $R$  down here. So I'll call this region  $R$ . And that region will be the projection of this solid region down. And it has some boundary curve--  $C$ , say-- the boundary curve of the region  $R$ . Just in case we need to refer to them later, it's good to give them letters so that they have names.

So what we need to figure out now is what is this region  $R$ ? Now this is tough to do by just intuitive reasoning or just by looking at this picture I've drawn. So in this case, we're kind of forced to use some algebra. All right.

So what do we know about this region  $R$  and this curve  $C$ ? Well,  $C$  is the projection downwards of the curve of intersection of this plane with this paraboloid, right? So it's the projection down at this curve intersection.

So what does that mean about its equation? Well, it means it's what we get if we solve for  $z$  in one of the two equations of the surfaces and plug it into the other. And that will give us an equation with just  $x$  and  $y$ , and that will be the equation of this curve  $C$ . OK.

So in our case, that means that  $C$  is given by this equation  $x^2 + y^2 = 2y$ . All right. So whenever  $x^2 + y^2 = 2y$ , that's a point  $x, y$  such that directly above that point is a place where the plane intersects the paraboloid. Well, OK.

So what is this curve? Well, a little bit of algebra can help us sort that out. If you bring the  $2y$  over here and complete the square, you can see that we can rewrite this as  $x^2 + (y - 1)^2 = 1$ . I brought the  $2y$  over, I've added 1 to both sides, and I've factored the  $y$  part. And so this is an easy equation to recognize.

This is the equation of a circle with center  $(0, 1)$  and radius 1. So let's draw that. And so here is a picture of what the shadow looks like in the  $xy$  plane. So the center is at height 1, and then it's this circle.

That's almost a circle. It looks enough like a circle for my purposes. So this is the region  $R$ . It's a circular region of radius 1 with center  $(0, 1)$ . OK, great. So I'm just going to shade that in again because I like doing that. OK. So that's the region  $R$ .

So what is this region  $R$ ? Let's look back over here. It's the shadow of our solid region in the  $xy$  plane. So when you project down, that's the region that you get.

So why do we need that? So we know when we set up this triple integral,  $z$  is going to be going from the paraboloid up to the plane. That's going to be the innermost integral, but then the middle integral is going to be  $y$  in terms of  $x$  or  $x$  in terms of  $y$ . Or if we do polar coordinates or cylindrical coordinates, it's going to be  $R$  in terms of  $\theta$ .

So we need to figure out what the boundary is, what that region looks like over which we'll be integrating for the outer two integrals. OK, so now I've been saying we could use cylindrical or we could use rectangular. What do we want to use?

Well, so this is a circle. It's not centered at the origin, but it is tangent to one of the axes at the origin. So this is a reasonably nice situation to do polar coordinates in, or cylindrical coordinates. You have to remember from when you learned cylindrical and polar coordinates what the equation of such a circle is.

And so I'm going to write it down here, and I'm going to invite you to go look up why this is true if you don't remember. This curve has an equation in polar-- these are the  $x$ - and  $y$ -axes here-- so this curve

has in polar coordinates the equation  $r = 2 \sin \theta$ . All right. So that gives me this curve here. The outer boundary.

And now what I want is, I don't just want the curve. I want to integrate over the whole region, and I want to integrate over it once. Remember, polar coordinates are a little tricky because you have to worry about are you overlapping and so on.

So how does this work? At  $\theta = 0$ , or at the origin, and then as  $\theta$  grows, we get further and further away. So this is our radius growing out. And then at  $\pi/2$ , we're at the top point of the circle. And then as it comes back into  $\pi$ , it comes back in.

So we want  $\theta$  going from 0 less than or equal to  $\theta$  less than or equal to  $\pi$  here. So at  $\pi/2$  at the top, and at  $\pi$  it comes back for the first time. And what about  $r$ ?

Well, it looks like  $r$  has to go all the way out to  $2 \sin \theta$ . And in fact, we always want it to start at the origin. So we always want  $r$  to go from 0 to-- this outer boundary--  $2 \sin \theta$ . So this describes this region  $R$  that we're trying to integrate over. This circular region in polar coordinates. So OK.

So it's a fairly easy description of polar coordinates. You could also describe it in rectangular coordinates, and you could try to solve the problem that way. I'm not going to do it for you, but you could give it a shot and see if you can come out with the same answer in the end that we do. So OK. So now, what have we done?

Well, I haven't written our bounds, so let me write our bounds on  $z$  right here. So we know that  $z$  is going from the paraboloid. If we look at the paraboloid  $z = x^2 + y^2$ -- but we're working in cylindrical coordinates now, so we need to write this in terms of  $r$  and  $\theta$ -- so that's  $z = r^2$ , and it's going up to the plane  $z = 2y$ -- now  $y$  in cylindrical coordinates is  $r \sin \theta$ . So  $z$  is going from  $r^2$  to  $2r \sin \theta$ .

So let's go write that down over here. So  $z$  is going from-- just ignore that-- from  $r^2$  less than or equal to  $z$ , and it's going all the way up to  $2r \sin \theta$ .

So these three equations describe our region. Yeah?  $0 < \theta < \pi$ : that just says  $\theta$ . OK? Then when  $\theta$  is going from  $0$  to  $\pi$ --  $r$  going from  $0$  to  $2 \sin \theta$ -- that says in the  $xy$  plane we're tracing out this circular shadow. And then as  $z$  goes from  $r^2$  to  $2r \sin \theta$ , that says above this shadow we're above the paraboloid and below the plane. So that's exactly the region that we want. So OK.

So now how do we get it to volume after we figured this out? Well, we write down the triple integral. So  $V$ , the volume of a region  $D$ , is equal to the triple integral over that solid of  $dV$ . OK? And in our case, in cylindrical coordinates,  $dV$  is going to be  $dz$  times  $r$   $dr$   $d\theta$ , or  $r dz dr d\theta$ . OK? So this is equal to, if we're integrating,  $r dz dr d\theta$ .

And now we need to put in our bounds. If we look over on this side of me, here they are. And these are our bounds that we're going to be using. So  $\theta$  is going from  $0$  to  $\pi$ . And  $r$  is going from  $0$  to  $2 \sin \theta$ . And  $z$  is going from  $r^2$  to  $2r \sin \theta$ .

So this triple integral gives us precisely the volume of our region. And in order to figure out what that volume is, we just have to evaluate this integral. So let's start doing that. I don't think I'm going to go quite all the way, but I'll do most of the work. So OK.

So let's do the innermost integral first. Whenever you have a triple integral like this-- a nice iterated integral-- you always start at the inside and work your way out. Well here, our integrand is  $r$ , and we're integrating with respect to  $z$ -- and  $r$  doesn't have any  $z$ 's in it-- so this inner integral is going to be easy. So I'm going to rewrite this as equal to-- we keep our outer two bounds-- so it's still from  $0$  to  $\pi$ , and it's still from  $0$  to  $2 \sin \theta$ , of  $2r^2 \sin \theta$  minus  $r^3$   $dr d\theta$ . So what I've done here is I've just integrated.

I get the anti-derivative of  $r dz$  is  $rz$ . And so then I take the difference between those two bounds. So I get  $r$  times  $2r \sin \theta$  minus  $r$  times  $r^2$ . So  $r$  times  $2r \sin \theta$  is  $2r^2 \sin \theta$ . Minus  $r$  times  $r^2$  is minus  $r^3$ . OK, so I've just done the first integral.

So now integrating with respect to  $r$ . OK, this second one isn't so bad either. As far as  $r$  is concerned, this is just a polynomial.  $\theta$  is constant with respect to  $r$  when we're doing an integral like this. So OK.

So the second integral is not too bad either. So this is the integral for our outer integral from 0 to  $\pi$ . Sticks around-- lets not do this one in one fell swoop I think-- so it's going to become  $2r^3 \sin^3 \theta$  over 4. And we're taking that between  $r$  equals 0 and  $r$  equals  $2 \sin \theta$ . And then that whole thing is going to be integrated  $d\theta$ .

So what do we get when we plug this in? Well, at  $r$  equals zero, this is just 0, so that's easy. And so we need the top one,  $r$  equals  $2 \sin \theta$ . So this is going to give me something like  $\frac{16}{3} \sin^4 \theta$  minus  $4 \sin^4 \theta$ , so I think that works out to be  $\frac{4}{3} \sin^4 \theta$ , between 0 and  $\pi$ .

So now you have to remember how to do integrals like this. So this is something you probably learned back in the trig integral section of your Calculus I or 18.01 class. So when it's an even power here, I think the thing that we do is we use our half-angle formulas.

So now I'm going to tell you what your final steps are. So first, you're going to use your half-angle formula. So what is that half-angle formula? So it's  $\sin^2 \theta$  is equal to  $\frac{1 - \cos 2\theta}{2}$ .

So you're going to have to plug this in here, right?  $\sin^4$  is  $\sin^2$  squared. And then you're going to have a  $\cos^2 2\theta$ , so you're going to have to use the double-angle formula. This time you're going to have to use the double-angle formula for cosine, which is very similar, although not exactly the same. So you're going to have to use those two double-angle formulas. After that, you'll have something that is straightforward to integrate.

So you'll have something that's straightforward to integrate. You'll integrate it, and if I'm not mistaken, what you get at the end is that you just get a fairly nice and simple  $\frac{\pi}{2}$  as your answer.

So you can check your work there, and make sure that you've got out  $\frac{\pi}{2}$  at the end. And hopefully, if you tried to do this using rectangular coordinates, you also came out with something like this as well. In that case, you would have to do a trig substitution at some point to evaluate your intervals, or you might have an arcsine involved. Something like that will happen. But it should also give you  $\frac{\pi}{2}$ , of course. Because it's the same region, just described in a different way.

So let me quickly recap what we did. Way back over here, we had this description of this region. So it was the region above our paraboloid and below a plane. And so when we're setting this up, we have to figure out in order to do a triple integral over this region in order to find its volume, we have to pick an order of integration, and then we have to know what the bounds are for the inside in terms of the outer two variables, for the middle one in terms of the outermost one, and so on.

So in this case, that means it was a natural choice to make  $z$  the first variable-- the innermost variable. And so then after that, we needed to project to find the relationship in the  $xy$  plane between the other variables. Now in this case, we did that by solving this little algebra problem here. We solved for  $z$  in the two surfaces that we were given, and we set them equal to each other.

And so this gives us a description for the boundary curve for our region. And because it's a nice circle, this suggested that one possibility was cylindrical coordinates. So we went ahead, and we found in cylindrical coordinates the description of this shadow. And then we used the knowledge we previously had to describe the whole region in cylindrical coordinates.

So we had this description of our entire region. And then to compute its volume, we just set up the triple integral volume is equal to a triple integral  $dV$ . In our case,  $dV$ -- since we're in cylindrical coordinates-- that's  $r \, dz \, dr \, d\theta$ . We put in our bounds, and then we evaluated the integral. I'll stop there.

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