

DAVID JORDAN: Hello, and welcome back to recitation. So today I want to practice doing, computing some integrals with you at polar coordinates. So we have these three integrals set up here.

And the way the integrals are given to you, they're given to you in rectangular coordinates. So the first thing you need to do is to re-express these in polar coordinates. And so for the first one, part a, I want you to completely compute the integral. For part b and c, we have this function f , which we haven't specified yet. So the exercise is to set up the integral. We won't actually compute it, we'll just set it up completely. So rewrite it in terms of r and θ .

So why don't you pause the video and get started on that. And check back with me and we'll work it out together.

Welcome back. Let's get started. So for all of these, when we're transferring from rectangular to polar coordinates, the most difficult part is understanding what region we're integrating over. And if we can understand that, then the rest of it is just straightforward calculation.

So in part a, let's try to think about what this region is that we're given. So we're given the region from x equals 1 to x equals 2. So let's draw those lines. x equals 1 and x equals 2. OK, so that's how x varies, in between this band.

And now the y varies from 0 to the function x . So we'd better draw the line y equals x . OK, so this is the line y equals x .

And the region that we're after is in between this band and in between these two lines, so it's this region right here. That's the region that we want. All right. So now that we have that, what we need to do to express something in polar coordinates is we need to sweep out rays of constant angle, and then measure the radius along the ray. So let's see what I mean by that.

So we can kind of see that if we're inside this region here, the smallest value that θ can be is just θ equals 0. Because, for instance right here is a point at θ equals 0. And then θ runs all the way up. And this whole line here is where θ reaches its maximum, and that's at θ is π over 4. So this tells us that it's easiest to write the θ integral on the outside. And we're going to have θ running from 0 to π over 4. OK, very good.

Now, in order to get the r ranges, it's going to be a little bit more subtle, because what we need to do is we need to fix some arbitrary θ in the middle here. So I'll just draw kind of a

representative theta. It would be like this one here. So there is a representative theta.

And what we need to do is, we need to describe how r varies along this pink line here. So that's the line that we really want. OK. So in order to do that, we just need to do some simple trigonometry now.

So let's see. So r turns on at this line right here. At the line x equals 1; that's where r turns on. So let's see what the value of r is at that line here. So, the point is that we have-- so let me draw this triangle here. Let me blow this triangle up. OK.

So this is our line x equals 1, and this is this triangle here. We just put a magnifying glass on it. So we have this angle theta. OK. So this x -value is 1, because we're on the line x equals 1. So the length of this leg is 1, and this is r that we're after. OK?

So in basic trigonometry we know that $\cos \theta$ is equal to-- so cosine is adjacent over hypotenuse-- is equal to 1 over r . OK, so that tells us that r is equal to $\sec \theta$. OK?

So what that means is that this, for any given theta, this pink band turns on at precisely r equals $\sec \theta$. And so that's what we need to write in the bottom here. OK? Now, you can see that it turns off at r equals $2 \sec \theta$. Because now, instead of going over length 1, we go over length 2.

And so our integral becomes the integral from 0 to $\pi/4$. And r equals $\sec \theta$ to $2 \sec \theta$. OK, and now we need to re-express our original function in terms of r and theta.

So our original function was-- so let's go back over to the formula here. So we have $x^2 + y^2$ to the $3/2$. So $x^2 + y^2$ to the $1/2$, that's r . And so $x^2 + y^2$ to the $3/2$, that's r^3 . So this is $1/r^3$.

So this is $1/r^3$ in here. And as we know, $dy dx$ becomes $r dr d\theta$. OK. So this is the integral that we need to compute, and now that we've done all the geometry, this is going to be a straightforward integral to compute. So let's do it together.

I'll just rewrite it over here. So we have theta running from 0 to $\pi/4$, and r running from $\sec \theta$ to $2 \sec \theta$. And we have $1/r^2 dr d\theta$. OK.

So that equals-- taking the inside integral first, we get $-1/r$ from $2 \sec \theta$ to $\sec \theta$. OK? And now this is nice, because $\sec \theta$ -- if we plug it in for $1/r$ -- it's just going

to turn back into a cosine, right? So this is just the integral from $\theta = 0$ to $\pi/4$.

OK, because we have this minus sign here, we're going to get $\cos \theta$ minus $1/2 \sec \theta$. And this we can compute to be the square root of $2/4$. So this is an elementary integral that we can compute: the square root of $2/4$. OK.

And that's it. That's all there is to a. So notice that the difficult part is figuring out how the boundary curves of your original region re-express in the r and θ coordinates. So let's see if we can get more practice with that on parts b and c.

So in part b-- let me just recall-- we're taking the integral $x = 0$ to 1 . And y equals x squared to x , $\int dy dx$. So let's draw this region again.

So the bottom curve is $y = x^2$ and the top curve is $y = x$, and x runs from 0 to 1 . So this is the region that we're after. OK.

So once again in this case, it's pretty easy to figure out the range for θ . Because you see this parabola here, as it approaches the origin, it becomes flat. So the θ , as we get closer and closer to the origin, is 0 .

So the initial bound for θ is 0 . It doesn't matter that we have this curve here, it's still 0 . And then the top bound, again, is $\pi/4$. OK?

Now, for this curve, $y = x^2$, what we just have to do is we just have to use our polar change of coordinates and re-express this curve. So $y = x^2$. Well, y is $r \sin \theta$. And x is $r \cos \theta$. So this is altogether $r^2 \cos^2 \theta$. OK?

And now all we need to do is just solve for r here. So we get r equals-- so I can cancel that r with that one-- and it looks to me like we get $r = \tan \theta \sec \theta$ by solving. OK. So that's the r -value at each of these points along the curve. So that is going to be our top bound.

So if we draw little segments of constant θ , they all start at $r = 0$, because we have this sharp point here at the origin. And they all stop at this curve: $r = \tan \theta \sec \theta$. All right?

And so now if we're just going to leave put. And then finally we have $r dr d\theta$. OK. So that's our answer to b. All right.

Now c is going to be a little bit tricky. What's tricky about c is drawing a picture, but we'll see

what we can do about that.

So let me just remind you. y ranges from 0 to 2, and x ranges from 0 to the square root of $2y$ minus y squared. And then we have $\int f \, dx \, dy$. OK.

Well, the y bounds are pretty straightforward, 0 to 2. This function, when I first saw this, I didn't recognize this function right away. So let's see what we can make out of this curve.

So the top curve for x is going to be x equals the square root of $2y$ minus y squared. And this looks like it wants to be the equation of a circle. So let's see if we can turn this into an equation of a circle.

So if we square both sides of this equation, then we get x squared equals $2y$ minus y squared. And now this over here, $2y$ minus y squared, so if we add this over to the other side, we have x squared plus y squared minus $2y$ equals 0.

And now the thing that I notice about this expression here is it really looks almost like the quantity y minus 1 squared. Yeah? If there were a plus 1 here, then that would be the case. So this is actually x squared, plus y minus 1 squared-- except the left-hand side is not quite equal to that, it's equal to that, minus 1. So this equation just equals 0.

So this equation is just equivalent to that one. And that's good news, because this says that, this is the equation if I add this 1 over to the other side-- let me go over here-- we have x squared, plus y minus 1 squared, equals 1. And this, we know what this is.

This is a circle which is centered not at the origin, but at the point 0 comma 1 in the y -direction. So if we draw this, well, this is the equation for the entire circle. But we only want the positive half, because we were told that x starts at 0 and goes up to this positive number. So we only want the positive half here. OK.

So this is the region that we're integrating. And now that we know this, we can again figure out the values of θ and the values of r . So first of all, θ is going to range again from 0, because this bottom curve, as it comes into the origin, it comes in flat. So we have points of arbitrarily small angles. So θ is going to start at 0, and θ is going to go all the way up until π over 2, because θ can be pointing straight up. OK?

And now we have to think about these lines of constant angle, and we need to think about what r does in there. So we can see from the picture that r always starts at 0 and it stops at

this curve. So we need to figure out what the r -value is along this curve in terms of θ . So what we want to do is use the equation.

So we had it over here. So $x^2 + y^2 - 2y = 0$. So that was one of the equivalent equations that we got for our curve. And now this is nice, because this here is just r^2 , isn't it?

So this is r^2 . And this is $-2y$, and y is $r \sin \theta$. OK. So we get $r^2 - 2r \sin \theta = 0$. Solving this for r , we get that r is just $2 \sin \theta$. OK.

So $r = 2 \sin \theta$. That is the equation in r, θ coordinates, which describes this semicircle here. And so altogether, we get the range is from $r = 0$ to $r = 2 \sin \theta$. And f just comes along for the ride, and we have $r \, dr \, d\theta$. And I'll leave it at that.