

DAVID JORDAN: Hello, and welcome back to recitation.

So in this problem, we're considering a function f of three variables, f of x , y , z , and it's differentiable. And we're not told a formula for f . We just know that it's differentiable at this point P , which is 1 , -1 , 1 , and we're told that the gradient of f at that point is this particular vector $2i$ plus j minus $3k$, at that point P . So all we understand about f is how it looks around the point P .

Now, we also have this relation between the variables, so x , y and z aren't unrelated. They're related by this constraint that z is x squared plus y plus 1 . So with this information, we want to compute the gradient of a new function g , and the new function g is a function of two variables, and this function g is obtained from f by just plugging in our relation for y . So we can use our constraint to solve for y , and then this function g is just f with that constraint applied. And what we really want to do is we want to find the gradient of g at the point $(1, 1)$. So notice that when g is equal to $1, 1$, that means that-- sorry, when the input of g is $1, 1$, that means the input of f is P . OK? So why don't you pause the video and work this out for yourself. Check back with me and we'll work it out together.

OK, welcome back. So let's get started. So this problem may not look like it's a problem about partial derivatives with constraints, but that's what it's really going to boil down to, which is to say that when we want to compute-- so when we want to answer this question by computing the gradient, the first thing we're going to want to do is compute the partial derivative of g and its variable x . And from the way we set up the problem, that's just the same as computing the partial derivative of f with respect to x and keeping z constant.

Now, remember, when we do partial derivatives with constraints, what's important about the notation is what's missing. The variable y is missing, and that's because we are going to use the constraint to get rid of it, and that's exactly how we define g , so this is the key observation. So computing the gradient of g is just going to be computing these partial derivatives with constraints. So we'll do that in a moment, and I'll also just write that the partial derivative of g in the z -direction is $\partial f / \partial z$, now keeping x constrained.

All right, so we need to compute these partial derivatives with constraints. And so you remember how this goes. The way that I prefer to do this is to compute the total differentials. So let's compute over here. The total differential df is-- the total differential of f is just the

partials of f in the x -direction, f in the y -direction, f in the z -direction, and each of these is multiplied by the corresponding differential. And we don't know f , so we can't compute the partial derivatives of it in general, but we do know these partial derivatives at this point. And so in the problem, we were given that this is $2dx$ plus dy minus $3dz$. So this is just using the fact that the gradient of f we were given is $2, 1$ minus 3 , OK?

So that's the total differential of f , and now we have this constraint. And remember, when we do these partial derivatives with constraints, the trick is to take the differential of the constraint. So we had this equation z equals x squared plus y plus 1 , and what we need to do is take its differential. So we have dz is $2x dx$ plus dy . So that's our constraint. Now here we have this variable x , but we're not varying x in this problem. We're only focused on the point P , and at the point P , x is 1 . So, in fact, dz is just $2dx$ plus dy , OK?

So now what we need to do is we need to combine the constraint equation and the total differential for f into one equation, and so this is just linear algebra now. So I'll just come over here. So we can rewrite our total differential for the constraint as saying that dy is equal to dz minus $2dx$, and then we can plug that back into our total differential for f . And so we get that df is equal to $2dx$ plus-- now I plug in dy here-- so dz minus $2dx$, and then finally, minus $3dz$. So altogether, I get a minus $2dz$, because this and this cancel. OK. We get a minus $2dz$.

So what does that tell us about the gradient? So remember that the differential now of g and its variables is $\text{partial } g \text{ partial } x dx$ plus $\text{partial } g \text{ partial } z dz$. And remember, the trick about partial derivatives and their relation to total differentials is that the partial derivative is just this coefficient. So the fact that we computed df and we found that it was minus $2dz$, that tells us that dg -- so the thing that we need to use is that g , remember, is just the specialization of f . So dg is equal to df in this case, because g is just a special case of f with constraints. So when we computed df here, we were imposing exactly the constraints that we used to define dg , and so what this tells us is that we can just compare the coefficients here and so our gradient is $0, \text{ minus } 2$. So the 0 is because there is no dependence anymore on x at this point. There wasn't a dx term. There could have been and there wasn't. And the minus 2 is because that's the dependence on z .

OK, so this is a bit complicated, so why don't we review what we did. So going back over to the problem statement, we first needed to realize that saying that g was a special case of f where we use our constraints to solve for y , that's what put us in the context of problems with constraints. So the fact that the dependence on y was so simple that we could just use the

constraint. OK.

So then, what that allowed us to do is it allowed us to say that the partial derivative of g in the x -direction is just the partial derivative in the x -direction subject to the constraint, and that's what we did here.

And so then, the next steps that we did are the same steps that we would always do to compute partial derivatives with constraints, and so we finally got a relationship that said that, subject to these constraints, df is equal to 0 . And the point is that g is just the function f with those constraints imposed, and so dg and df are the same, and so, in particular, dg is 0 . And then, we remember that you can always read off partial derivatives from the total differential. They're just the coefficients. And so in the end, we got that the gradient was 0 . And I think I'll leave it at that.