

## Integrals in Spherical Coordinates

1. Find the volume of a sphere of radius  $a$ .

**Answer:** From the problems on limits in spherical coordinates (Session 76), we have

limits: inner  $\rho$ : 0 to  $a$  –radial segments

middle  $\phi$ : 0 to  $\pi$  –fan of rays.

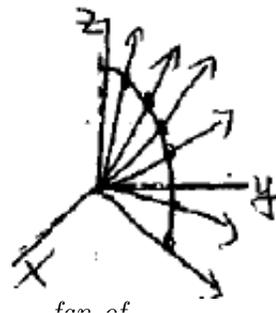
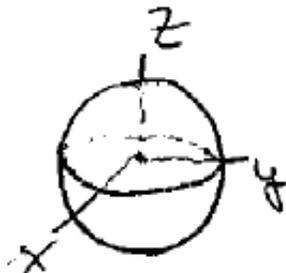
outer  $\theta$ : 0 to  $2\pi$  –volume.

$$V = \iiint_D dV = \int_0^{2\pi} \int_0^\pi \int_0^a \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$$

$$\text{Inner: } \frac{\rho^3}{3} \sin \phi \Big|_0^a = \frac{a^3}{3} \sin \phi$$

$$\text{Middle: } -\frac{a^3}{3} \cos \phi \Big|_0^\pi = \frac{2}{3} a^3$$

$$\text{Outer: } \frac{4}{3} \pi a^3 \text{ –as it should be.}$$



2. Find the centroid of the region bounded by the sphere  $\rho = a$  and the cone  $\phi = \alpha$ .

**Answer:** In Session 76 we computed the limits:

$\rho$ : 0 to  $a$ ,  $\phi$ : 0 to  $\alpha$ ,  $\theta$ : 0 to  $2\pi$ .

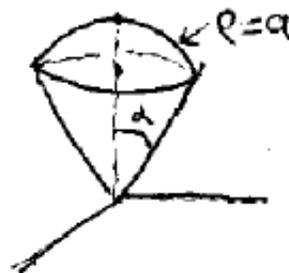
By symmetry,  $x_{cm} = y_{cm} = 0$ .

$$\begin{aligned} z_{cm} &= \frac{1}{V} \iiint_D z \, dV = \frac{1}{V} \int_0^{2\pi} \int_0^\alpha \int_0^a \rho \cos \phi \cdot \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta \\ &= \frac{1}{V} \int_0^{2\pi} \int_0^\alpha \int_0^a \rho^3 \cos \phi \sin \phi \, d\rho \, d\phi \, d\theta. \end{aligned}$$

Inner, middle and outer integrals are easy to compute:  $z_{cm} = \frac{1}{V} \cdot \frac{\pi a^4 \sin^2 \alpha}{4}$ .

$$V = \iiint_D dV = \int_0^{2\pi} \int_0^\alpha \int_0^a \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta = \frac{2}{3} \pi a^3 (1 - \cos \alpha).$$

$$\Rightarrow z_{cm} = \frac{a^4 \sin^2 \alpha \pi}{4} \cdot \frac{3}{2\pi a^3 (1 - \cos \alpha)} = \frac{3a}{8} \cdot \frac{\sin^2 \alpha}{1 - \cos \alpha} = \frac{3}{8} a (1 + \cos \alpha).$$



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