CHRISTINE BREINER: Welcome back to recitation. In this video, we want to work on using the change of variables technique. In particular, we're going to look at the following problem.

It says, using the change of variables, u is equal to x squared minus y squared, and v is equal to y divided by x. Supply the limits and integrand for the following integral, which is the double integral over region R of 1 over x squared, dxdy. And R is the infinite region in the first quadrant that is both under the curve y equals 1 over x, and to the right of the curve x squared minus y squared equals 1.

So this is a challenging problem. Again, I want to point out we just want to find the limits and the integrand. You don't actually have to compute the integral. But it is going to be tough, but stick with it. Pause the video, give it your best shot-- hopefully you find the appropriate limits and integrand-- and then when you feel comfortable, bring the video back up, and I'll show you how I do it.

OK. Welcome back. So once again, what we want to do is this change of variables problem where we've defined u to be x squared minus y squared, v to be y divided by x, and we have this region that is in the first quadrant and it's under the curve y equals 1 divided by x and it's to the right of the curve x squared minus y squared equals 1. And we want to compute the limits and integrand for that particular integral.

So what I'm going to do to try and make this as organized as possible is I'm going to try, first to graph the region R-- to figure out what the region R is in the xy plane. Then I'm going to try and figure out what the region R is mapped to in the uv plane. So what it looks like in the uv plane. That will give me my limits.

And then I'm going to try and determine the Jacobian. And then I will determine from that and the fact that I started with 1 divided by x squared as my function I was integrating, I will put those two together to figure out the integrand. So there are a bunch of steps to these problems. But the first one, again is I'm going to graph the region R.

So I'm going to give you a very rough sketch over here of the region R. And I know it's in the first quadrant and I know it's infinite. I was already told that. OK. So in the xy plane, the region R is below the curve y equals 1 over x. So let me draw that curve.

Again, this is very rough. This is a rough sketch. I'm putting up no scale on purpose. I'll put in one value, maybe, in this whole thing. OK?

And so this is the curve y equals one over x. And then I need the curve which is part of the hyperbola that is x squared minus y squared equals 1. So I'll draw in the part we need, which looks roughly like this. Something like that. Again, this is not meant to be an exact graph, but to give you an idea of what the region looks like.

And the only thing I'm going to mention is that this point we know is x equals 1 and y equals 0. So the region we're interested in that is both to the right of x squared minus y squared equals 1 and below y equals 1 over x and in the first quadrant is exactly this region I'm shading here. So we want to understand what the values of u and v are along these bounds. We need to understand where this region maps to when I do the change of variables in order to understand what the limits are. So let me put the graph of this region in the uv plane so that we can really understand what our bounds are.

And I know already where it's going. So I'm going to just make the first quadrant, because I know this is going into the first quadrant. So it doesn't always work that something in the first quadrant maps into the first quadrant, but in this case, I already did the work, so I know it does. So let me point out a few things about where this region R maps.

The first thing I want to point out is that we actually know that this curve under the change of variables maps to u equals 1. Because if you remember, u is equal to x squared minus y squared. So this whole curve is going to map to u equals 1.

Now, I don't want the whole curve for my region. I only want this little piece of it. So I'm going to have-- in my uv plane-- I'm going to have some segment at 1. And actually, I'll just know that it's some part of the line u equals 1 is going to show up in there.

But if you notice, I know where it starts right away. Because at x equal 1, y equals zero, if I look at what v is-- if we come back here and remember what v is-- at x equal 1, y equals 0-- v is 0. And so my starting point on this segment --if we come back here-- my starting point on this segment is actually also at 1, 0. OK?

So I know there's some point right here that maps down to here where the segment will stop. I'll find that point later, algebraically. Right? And then now we need to figure out where these two curves go. And then we can get a picture, and then we'll figure out what that point is, and we'll understand all the limits.

So the first thing I want to point out is along this curve, we have y equals 0 and x is non-zero. And just to help ourselves, I'm going to rewrite what the change of variables is here, so I don't have to keep walking over to the other side. Our change of variables was u is equal to x squared minus y squared, and v was equal to y divided by x.

So this whole curve has y equals 0. So what happens to u and what happens to v along that curve? Well, u is going to be x squared, and v is going to equal 0.

And so the point of this, really, is-- that even though in u, this curve maybe is mapping at a different speed in some form to this curve here-- it still it's just taking that segment goes, or that infinitely long ray goes to an infinitely long ray here along the u axis. And again, that's because along this ray, y equals 0. And so v is equal to 0 everywhere on that ray and u is positive-- it's equal to x squared. OK? So I'm going to move the u out of the way, because we're going to say this is part of the region, or that's one bound of the region.

And now I have to figure out where this curve goes. This curve is slightly more complicated, but I can still figure it out. So I'm going to show you how I do that sort of algebraically.

That curve-- if you notice, if you remember-- is y equals 1 divided by x. So that means that on that curve-- let me even write it down-- on y equals 1 divided by x, v is equal to 1 divided by x divided by x. So v is equal to 1 divided by x squared, right?

And then what does that mean about u? u, then, is equal to-- well, x squared is 1 divided by v, and then y squared, because y squared on that curve is just 1 divided by x squared, is v. So let me just make sure we all followed that one more time.

We're looking at where the curve y equals 1 over x goes in the change of variables, right? So that's the top curve up here. y equals 1 over x is the top curve of our region R. So we want to know where that goes.

Well, on y equals 1 over x, v is exactly equal to 1 over x squared, because v-- we know-- is y over x. So if I just substitute in for y, I get 1 over x squared. Now, if I look at this relationship, this means x squared is equal to 1 over v. So in terms of u, x squared becomes 1 over v. And then y squared-- which is 1 over x squared-- become v. So that curve is u equals 1 over v minus v.

Now that curve, roughly, is going to look something like this. And it might seem strange. The thing is, I'm graphing this in the uv plane, and I'm writing what looks like u as a function of v, and so it's sort of turned around from how you usually see these things written. But this is the equation that describes this curve.

And that is sufficient to understand, because when we use, determine our bounds, we can determine our bounds from u equals 0 now, to u equals 1 over v minus v. So we now have the bounds in u. We're actually doing quite well.

So we have this region. We now have the bounds completely in u. u is going from u equals 0 to u equals 1 over v minus v. But the problem is now we don't know the bounds in v. We don't know what the bounds are in v, and so we have to be a little bit careful.

So actually, no. I think I was wrong. It's not 0, is it? I said that twice now, and that was incorrect. u is going from 1, to 1 over v minus v. So I apologize.

Because the slices of u are going from whatever the u-value starts-- which is at the value 1-- and it's coming this way. So I apologize. I was moving my arm like I was doing the v-values, but I actually wanted to do the u-values. So I want to go from where u starts-- which is at u equals 1-- to where u stops-- which is when it hits the curve 1 over v minus v equals u.

So hopefully I didn't confuse anyone by that. I'm glad I caught it, then. OK, so now we understand the bounds in u.

And then to understand the bounds in v, all we need to understand is what is the v-value at this point. So once I know the v-value at this point, then I'm done with the bounds. So let's see if we can find that.

Well, the v-value at that point is going to be at the point where these two curves intersect. So let's see if we can do a little algebra to understand what that will look like. So let me point out that where those curves intersect, I have the equation x squared minus 1 over x squared is equal to 1. And if I want to find x-values that satisfy this, I'm also looking for x-values that satisfy x to the fourth minus 1 is equal to x squared, which I can rewrite as x to the fourth minus x squared minus 1 is equal to 0. So I can actually use the quadratic formula on this in terms of x squared.

So what I get is I get x squared is equal to 1-- I get plus or minus root 5-- over 2. And if you look at it, the one you're actually interested in-- you can figure this out pretty quickly-- is the one that is plus. OK? I want the one that is plus root 5 over 2.

So then that means x is the square root of this quantity at that point, right? Or I could actually think about it this way. Let me point out this. v is equal to 1 over x squared at that point, because it's on that curve where we were talking about y equals 1 over x. So v is 1 over x squared.

So 1 over x squared is just 1 over this quantity. So it's the reciprocal of this. It's also negative 1 plus root 5, over 2. You can check that if you need to. But I will write it down this way as the following: this is the point 1 comma a. And if I come over here, I will denote a will equal negative 1 plus root 5, over 2. And that's really just 1 divided by x squared. So let me point that out again, that a is equal to 1 divided by x squared at the point of intersection.

So hopefully you can see all that. So that tells us our bounds completely. We still have some work to do.

So I'm going to put in the bounds and I'm going to leave an empty space. Actually, no. I won't do that, because this can get a little messy and confusing. So I'm just going to do the Jacobian, and then we'll figure it all out and write the answer right at the end, so there's no confusion.

But hopefully you see at this point that we have the bounds. We know that u goes from 1, to 1 over v minus v. And v goes from 0 up to a, where a is the value I've written here. So we know the bounds. So now we have to figure out the integrand.

So let's first compute the Jacobian, OK? So now we're looking at del u, v, del x, y using the notation we've seen in class. And so del uv, del xy is going to be the determinant of the following matrix: 2x, negative 2y. And then the derivative respect to v of x is negative y over x squared. And the derivative of v with respect to y is just 1 over x.

So if I take the determinant of that, I get 2, minus 2, y squared over x squared. Which if you notice, in terms of our change of variables, is exactly equal to 2 minus 2v squared, because v is y over x. And so I can rewrite this as 2 times the quantity 1 minus v squared. OK? So, so far so good, hopefully.

Now let's figure out how to do the final step. So the final step-- I'm going to come back over and just remind us what the integrand was, OK? If we come over here, we're reminded that we were integrating over the region R of 1 over x squared, dxdy. Right? That's what we were interested in initially. So now, if we come back, I'm going to write that down just to have it as a reference.

OK, that's what we had initially. Let me make sure. Yes, that's what we had initially. And so now we know dxdy is equal to dudv over 2 times 1 minus v squared. So that is going to replace the dxdy. And now we have to figure out what to do with the 1 over x squared.

But, what do we have here? Now I have to remind myself. I can't remember all the steps anymore. We have u is equal to x squared minus y squared. Let me come back. Now I've forgotten what I was doing. Ah, yes. Now I remember, sorry. OK.

So the point I should have remembered that I forgot, is that 1 minus v squared is equal to u divided by x squared. That's what I had figured out earlier that I just forgot when I was staring at the board. And to notice that, what do we have to remember? u is x squared minus y squared, so if I divide everything by x squared, the first term is 1 and the second term is v squared.

So, whew, that's good. So I was a little nervous there for second, but I did in fact do this earlier. And I'd forgotten what I did.

So now, the 1 minus v squared is actually the same as u divided by x squared. And notice what that does to this term here. That tells us that dxdy over x squared is actually equal to dudv over-- instead of the 1 minus v squared, I put u over x squared and I get-- notice, I get-- an x squared times 2, u divided by x squared. Right? I just replace the 1 minus v squared with what I know it is, the x squareds divide out, and so I get dudv over 2u.

So now the good news is I have all the pieces, because I'm about to run out of board space. So I have all the pieces, so I'm just going to put them together, and then we're done. So let me come here in the final spot, and say this is our final answer.

Our final answer is that we're integrating u from 1, to 1 over v minus v. And then we're integrating v from 0 to a-- where a is the value I determined earlier-- of 1 over 2u, dudv. So this is the final, final answer.

This was a long one. And I'm sorry I had a little brain freeze in the middle. I couldn't remember how I'd fixed that problem. So what I did at that point-- I just want to point out-- that when I was working on this problem, and I had a 1 minus v squared, I knew somehow I had to figure out how to relate that and the x squared in terms of u and v.

And so I actually saw this expression. I could have written it better, maybe, as x squared times this equals u. OK. And maybe that would have been more obvious, if that's the case. But that was really the step that allowed me to replace all of this by things in terms of u and v. Which I know I should have been able to do, it's just a matter of figuring it out. So let me just go back to the beginning and remind you of each of the steps very briefly, and then we'll be done. So we come back over to the beginning. We were starting with change of variables supplied for us. We already had an integral in terms of x and y, and we had an infinite region. And what we were asked to do is find the limits and the integrand.

So the first step for me is I always find it very helpful to draw the region in the xy plane, and then draw the new region in the uv plane. Neither one of them has to be perfect, but the understanding of the values of the curves in terms of equations of u and v are very important, to understand that. That gives you the bounds, the limits.

And then, so we did all this work. We found the limits. There was a little algebra in the middle. We found the limits. And then we found the Jacobian, which was going to tell us how the variables were changing. We found it in terms of x and y. We rewrote it in terms of u and v.

And so when we came back and we compared what our integrand was initially, we could compare dxdy to dudv. But then we also had to figure out how to replace the 1 over x squared. So once we did all that, we had everything in terms of u and v, and then we finally had what the integrand was going to be. So there were a lot of steps, but this was ultimately what the problem was.

And again, I'll just point out, this is the final solution right here. We integrated from 1, to 1 over v minus v, for u. And we integrated from 0 to a in v, the function 1 over 2u. OK. That is where I will stop.

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Fall 2010