18.02SC Multivariable Calculus, Fall 2010
Transcript – Recitation 60, Flux and the Divergence Theorem

JOEL LEWIS: Hi. Welcome back to recitation. In lecture, you've been learning about flux and surface integrals in the divergence theorem, and I have a nice problem about that for you here.

So I've got this field F, and it's a little bit ugly right? All right. So its coordinates are x to the fourth y, minus 2x cubed y squared, and z squared. And it's passing through the surface of a solid that's bounded by the plane z equals 0, by the plane z equals h, and by the surface x squared plus y squared equals R squared. So often we call this solid a cylinder.

So it's got its bottom surface in the plane z equals 0, and its top surface in the plane z equals h, and it's got a circular base with radius R there. So what I'd like you to do is to compute the flux of this field F through this cylinder.

So I'll point out before I let you at it, that to compute this as a surface integral, you could do it. You could do it. If you really want an exercise in nasty arithmetic, I invite you to do it. But you might be able to think of a way to do this that requires less effort than parametrizing the three surfaces and integrating and so on. So I'll leave you with that. Why don't you pause the video, work this one out, come back, and we can work on it together.

Hopefully, you had some luck working on this problem. Right before I left, I mentioned that you were computing a flux through a surface here, but that doing it as a surface integral is maybe not the best way to go. And so, even without that hint, probably many of you realized that really the way that we want to go about this problem is with the divergence theorem. OK.

So in our case, the divergence theorem-- I'm just abbreviating it div T-H-M here-- says start with the double integral over the surface of F dot n d surface area. So S here is the surface of this solid. So the divergence theorem says that this surface integral, which is the flux that we're interested in, is equal to the triple integral over the solid region D-- so that's bounded by the surface, and so that's the solid cylinder here-- of div F dV. OK.

So in our case, this is nice, because in fact, this solid region D is an easier to understand, or easier to grapple with region than the surface that we started with, right? It's just one solid piece. It's easy to parametrize, in fact. It's easy to describe especially in cylindrical coordinates, but also in rectangular coordinates. Whereas this surface S, if we wanted to talk about it, we'd need to split it up into three pieces, and we'd need to parametrize it. And it's kind of a hassle, relatively speaking.

Also, the divergence of this field F is a lot simpler than the field itself. If we go and look at this field, all of its components are polynomials. To compute its divergence, we take derivatives of all of them. And so that makes their degrees lower, and then we just add them. Life is a little bit simpler. So OK.

So this process of using the divergence theorem is going to make our lives easier. It's going to make this nasty surface integral into an easy to compute triple integral. So let's see actually how it does. So let's compute div F first. So we know what the integrand is. All right.

So we need to look at the components of F, and so we need to take the partial of the first one with respect to x. So that's x to the fourth y with respect to x. So put that down over here. That's 4x cubed y. We just treat y as a constant.

OK, so now we come back and we need to look at the second one. So it's minus 2x cubed y squared. And it's the second one. We take its partial with respect to y. So OK. So that's going to be minus 2x cubed times 2y. So that's going to be minus 4x cubed y.

And then we come back and we look at the last component. And that's z squared. And so we need to take its partial with respect to z. So in this case, that's just 2z, and so we add that on as well. Plus 2z.

And in this case, not only are these polynomials simpler than the coordinates of F that we had, but in fact, we've got some simplification here. Life gets really, really simple. So in fact, this is just going to work out to 2z. So the divergence here is very simple compared with the function F. More simple than we have a right to expect, but in any case, good. It's nice to work with. OK. So that's the divergence.

So I'm going to write, this is the flux. These integrals that we're interested in. This surface integral, and then by the divergence theorem, it's the same as this triple integral. So the divergence is this 2z.

So the flux is what I get when I just put that in here. So flux is equal to the triple integral over our solid of 2z dV. OK, so I've left some stuff out of this. Because I'm going to start writing down the bounds and writing this down as an iterated integral now. OK.

So we have to choose some coordinate system in which to integrate over this solid. And so we have three kinds of natural choices that we always look back to. There are rectangular coordinates and cylindrical coordinates and spherical coordinates. So spherical coordinates seem pretty clearly inappropriate. Rectangular and cylindrical? You know, you could try and do it in rectangular. It's not horrible. But this is a cylinder, right? I mean, it's crying out for us to use cylindrical coordinates. So let's use cylindrical coordinates.

So we're going to use cylindrical coordinates. So to get dV we need a z, an r, and a theta, but remember there's this extra factor of r. So it's going to be 2z times r dz dr d theta. Right? This is dV. This r dz dr d theta part. So that's what dV is when we use cylindrical coordinates.

OK, so now let's figure out what the bounds are. So let's go look at the cylinder that we had over here. So it's bounded between z equals 0 at the bottom surface and z equals h at the top surface. OK. So that's easy enough. That's what the bounds on z are. So let's put those in. So z is the innermost one, so that's going from 0 to h. OK.

How about the next one? So the next one is r. So let's go back over here. So r is the radius here after we project it down. And we just get the circle of radius big R centered at the origin. So little r is going from 0 to big R.

And theta is the circle. It's the whole circle. So theta is going from 0 to 2 pi.

So cylinders are really easy to describe what they look like in cylindrical coordinates. So let's put those in. So little r is going from 0 to big R, and theta is going from 0 to 2 pi. OK. Wonderful.

Now we just have to compute this, right? We've got our flux is this triple integral. So let's compute it. Let's walk over to this little bit of empty board space.

OK, so we have an iterated integral. So let's do it. So the inner integral is the integral from 0 to h of 2zr dz. Well, that's not that bad. That's equal to r as a constant. So it's equal to rz squared as z goes between 0 and h. It's dz, so z is going from 0 to h.

So plug in, and we just get h squared r minus 0. So just h squared r. OK. So now let's do the middle integral.

So the middle integral is the integral from 0 to big R d little r of the inner integral. So this is the integral from 0 to big R of the inner integral which was h squared little r d little r. OK. And that's not that bad either. So h is just a constant. So this is equal to 1/2 h squared r squared from r equals 0 to big R. And so that's 1/2 h squared big R squared. That's the middle integral.

So the outer one now. OK. So let's go back and look. So we're doing d theta as theta goes from 0 to 2 pi of whatever the middle integral was. So it's the integral from 0 to 2 pi of whatever the value of the middle integral was. So this is 1/2 h squared big R squared d theta. And this is all just constant with respect to theta. So that's going to be just pi h squared r squared. You're just multiplying it by 2 pi. All right. So pi h squared r squared. So this is our final answer.

Let's just quickly recap what we did. We had to compute the flux of this field F through the surface of a solid cylinder. And so we had options. We could do it directly by trying to compute the surface integrals, but in this case, life was a lot easier if we applied the divergence theorem. So the divergence theorem says that the flux-- which is equal to this surface integral-- can also be written as the triple integral over the solid region bounded by the surface of the divergence of the field. All right.

And so in our case, the divergence was very nice and simple, and the solid region D was relatively simpler to describe than-- its surface that bounds it-- S. So this is why we think of the divergence theorem. Because the divergence of the field is easy to understand, and the solid is easier to describe than its surface. So those are both things that make us think to use the divergence theorem for a problem like this.

So then by the divergence theorem, the flux is just that triple integral, and so we wrote it out here that we were integrating over a cylinder. So a natural thing to do is use cylindrical coordinates. And then we computed the triple integral just like we always do. I did it in three steps: inner, middle, and outer. You don't have to do it exactly this way if you don't want to. But it works for me. OK. And we got our final answer: pi h squared r squared.

I'll stop there.