

**JOEL LEWIS:** Hi. Welcome back to recitation. In lecture, you've been learning about the divergence theorem, also known as Gauss's theorem, and flux, and all that good stuff. So I have a nice exercise on it for you here.

So what I want-- so I want you to take  $F$ , and I want it to be the field whose components are  $x$  over  $\rho$  cubed,  $y$  over  $\rho$  cubed, and  $z$  over  $\rho$  cubed. So here,  $\rho$  is your usual  $\rho$  from spherical coordinates.  $\rho$  is equal to the square root of  $x$  squared plus  $y$  squared plus  $z$  squared. And I want  $S$  to be the surface of the box whose vertices are plus or minus 2, plus or minus 2, plus or minus 2. So it's a cubical box.

So what I'd like you to do is, first in part a, I'd like you to show that the divergence of  $F$  is 0, wherever the field  $F$  is defined. In part b, what I'd like you to think about is whether we can conclude from that, that the flux through the surface of  $S$  is equal to 0. All right. And in part c, what I'd like you to do is to use the extended version of Gauss's theorem-- or the extended version of the divergence theorem-- in order to actually compute the flux through  $S$  by computing an integral.

So why don't you pause the video for a couple of minutes, work out this problem, come back, and we can work it out together.

Hopefully you had some luck with this problem. Let's get started. Part a asks you to compute the divergence of  $F$ . So in order to compute that, we're going to need to take the partial derivatives of the components of  $F$ . And in order to do that, at some point I'm going to need to take a partial derivative of  $\rho$ . So let me first compute the partial derivatives of  $\rho$ , and that will save me a tiny bit of work later.

So  $\rho$  is equal to the square root of  $x$  squared plus  $y$  squared plus  $z$  squared. So partial  $\rho$  partial  $x$ -- well, you just apply your usual chain rule here. And I guess we get a half, but then we get a 2 that cancels it, so I think this works out to  $x$  divided by the square root of  $x$  squared plus  $y$  squared plus  $z$  squared, so that's  $x$  divided by  $\rho$ . All right. And I'm just going to keep  $\rho$  around here, because otherwise I have to write out the square root of  $x$  squared plus  $y$  squared plus  $z$  squared over and over again, and this is going to save me some effort and would save you some effort as well.

So OK. So this is  $\rho$ . So this is  $d\rho/dx$ .

So we want to take the  $x$  partial of the first component of  $F$ . So that's the  $x$  partial of  $x$  over  $\rho$  cubed. OK. And you just apply your usual quotient rule, so what do we get? We get the derivative of the top.

So that's  $\rho$  cubed minus-- OK, so the top is  $x$  times the derivative of the bottom, which is going to be  $3\rho$  squared times  $x$  over  $\rho$ -- so that's  $3$ -- so we have an  $x$ -- so it's  $3x$  squared  $\rho$ , divided by the bottom squared, which is  $\rho$  to the sixth. And I guess there's a common factor of  $\rho$  everywhere that we can cancel out. So this is equal to  $\rho$  squared minus  $3x$  squared divided by  $\rho$  to the fifth. OK. So that's the  $x$ -partial derivative of the first component of  $F$ .

Now we need the  $y$ -partial derivative of the second component of  $F$ , and the  $z$ -partial derivative of the third component of  $F$ . But if you go and look back at what the formula for  $F$  was, you see that this is a very, very symmetric formula. So in order to get from the first component to the second component, we just change  $x$  to  $y$ , and to get from the second component to the third, we just change  $y$  to  $z$ , because of course  $\rho$  treats  $x$ ,  $y$ , and  $z$  the same. So what does that mean? Well, that means that the partial derivatives are easy to compute.

Having computed this  $x$ -partial derivative, we also get that partial over partial  $y$  of the second component-- which is  $y$  over  $\rho$  cubed-- is equal to  $\rho$  squared minus  $3y$  squared, over  $\rho$  to the fifth. And the last one we get, partial over partial  $z$  of  $z$  over  $\rho$  cubed is equal to  $\rho$  squared minus-- I'm getting a little cramped here--  $3z$  squared, over  $\rho$  to the fifth.

And so adding these up, we get that  $\text{div } F$  is equal to the sum of those three things. So let's see what we've got. We've got a  $3$ -- so the denominators are all  $\rho$  to the fifth. And we've got  $3\rho$  squared minus  $3x$  squared minus  $3y$  squared minus  $3z$  squared. So this is equal to  $3\rho$  squared minus  $3x$  squared minus  $3y$  squared minus  $3z$  squared, all over  $\rho$  to the fifth.

But of course,  $\rho$  squared is  $x$  squared plus  $y$  squared plus  $z$  squared, so this numerator is just  $0$ . So this is equal to  $0$ . OK. Which is what we thought it should be. All right. Good.

So that's part a. We just computed the partial derivatives of  $F$ , and then added them together to get the divergence. And we found that, in fact, yes, the divergence was equal to  $0$ . Great. So that's part a. So let's go look at what part b was.

Part b asks, can we conclude that the flux through the surface  $S$  is  $0$ ? All right. Now remember

what the divergence theorem says. The divergence theorem says that the flux through a surface of a field is equal to the triple integral of the divergence of that field over the interior, provided the field is defined and differentiable and nice, or whatever, Everywhere inside. OK?

But this field has a problem. Almost everywhere, this field is nicely behaved, but at 0, we have a real problem. We're dividing by 0. Right? So this field is not defined at 0. So there's a single point in the middle of this cube where this field behaves badly. And that means we can't apply the divergence theorem inside this cube.

So since we can't apply the divergence theorem, we aren't allowed to conclude immediately that the flux through this surface is 0. OK. So the answer is no. We can't conclude that the flux through  $S$  is 0, because one of the hypotheses of the divergence theorem isn't satisfied. Namely, the field isn't defined everywhere inside the surface.

OK. So the answer to b is no. OK, I'm just going to write that. But it's no because the hypotheses aren't satisfied.

OK, so now let's look at part c. So part c suggests, we can't conclude that the flux is 0. So we still want to know what the flux is. That's still an interesting question, so part c suggests, maybe you can still use the divergence theorem-- well, now we're calling it extended Gauss's theorem-- to compute what this flux is. So let's think about how we could do that.

So remember what extended Gauss's theorem says? Or extended divergence theorem. I'm going to try and just say Gauss's theorem from now on, so I stop having to say both. But I mean both. I mean, they're the same theorem, right? OK.

So Gauss's theorem says, when you have a surface bounding a region, the flux through the surface is equal to the triple integral of divergence over the region, provided everything is well-defined and nice. Extended Gauss's theorem says, this is still true if your region has more than one boundary.

So for example, if your region is a hollow something-- so if it's a spherical shell that has an outside sphere and an inside sphere-- then extended Gauss' theorem says, OK, so you do the same thing. You take the triple integral of the divergence over the solid region. And then you take the flux, but you add up the flux over all of the boundary pieces.

So you add up the flux over the outside boundary surface, and also, if there is one, through any other boundary surface. OK? And those two things are equal. So the total flux through all

of the boundary surface is equal to the integral of divergence over the whole region bounded by those surfaces. So how are we going to use this?

We're trying to compute the flux through a surface. OK, but we don't want to compute a double integral if we can avoid it. We don't want to compute the surface integral. So what we'd like to do is we'd like to find a convenient region over which to compute this integral, to put us in a situation where we can apply extended Gauss's theorem. We can't use just the inside of the cube, so we want some other region.

So what we're going to do is we're going to walk over here, and we're going to do-- there are many possible things you could do, but this is a nice one. All right. One thing you could do is you could take a big sphere.

Take a big sphere. So we've got our cube here. This is the point  $(2, 2, 2)$ , and this is the point  $2, 2, 2$ , minus 2, and so on. So we've taken a big sphere of radius  $R$ -- for some big  $R$ -- that contains our surface  $S$  that we're interested in, that completely contains the cube. OK? So why have we done that? Well, extended Gauss's theorem--

OK, so what does extended Gauss' theorem say for the region between the sphere and this cube. All right. So our cube is named  $S$ . Let's call our sphere  $S_2$ , because why not? OK. And let's call the solid region between them, between the cube and sphere-- just for convenience, let's give it a name-- so, I don't know, we often call solid regions  $D$ , so let's call it  $D$ .

So it's this spherical region, but it has a cubical hole in the middle of it. OK. So what does extended Gauss's theorem say? So extended Gauss's theorem says that the triple integral over  $D$  of the divergence of  $F$   $dV$  is equal to the sum of the fluxes through each of the surfaces. But for this, we want the flux out of the solid region.

So for the sphere, the flux out of the inside of the sphere is the flux out of the sphere. So that's integral over  $S_2$  of  $F \cdot n$ ,  $d$  surface area. But for the cube, the flux out of this region is the flux into the cube. Right? Out here, you're living in a region outside the cube, so when you leave that region, you're going into the cube. So this is the negative of the flux that we really want.

So this is minus the flux through the cube of  $F \cdot n$ , with respect to surface area. So remember, the signs here are different, because I'm taking this normal to be the outward pointing normal to both surfaces. The normal that points away from the origin. But the normal

pointing away from the origin on the cube is the normal that points into the solid region instead of the normal that points out of the solid region. So that's why this minus is here. OK. Whew.

All right, so what does this mean? Well, we've already computed, in part a, that the divergence-- so first of all,  $F$  is well-defined everywhere in this region  $D$ . The only place  $F$  was badly behaved was the origin. And this region doesn't contain it, which is why this trick works. So we've already computed that the divergence of  $F$  is 0 everywhere. It's defined, so it's 0 on all of  $D$ , and so this triple integral is just 0.

So if this triple integral is 0, that means we can just add the thing that we're interested in to both sides, and we get that the surface integral over the cube of  $F \cdot n$ , with respect to surface area, is equal to the surface integral over the sphere of  $F \cdot n$ , with respect to surface area. OK.

So we've converted this original integral-- our flux integral that we're interested in-- and we found that it's equal to this separate flux integral over a different surface. This time over a big sphere. OK, so that's nice. Why do we want to do that? Well, we want to do that because  $F$  is a really nicely behaved field with respect to a sphere.

$F$  is a radial field. So  $F \cdot n$  is really easy to understand.  $F \cdot$

$n$  is just-- well,  $n$  is a unit normal and  $F$  is a radial field. So on a sphere, the normal is radial, right? It's parallel to the position vector. And  $F$  is radial. So they're both pointing in exactly the same direction.

So when you take that dot product,  $n$  is the unit vector in the same direction as  $F$ , so when you dot that with  $F$ , you just get the length of  $F$ . OK, so what does that mean? That means over here, this integrand is really easy to understand. OK? This integrand  $F \cdot n$  on the sphere is just equal to the length of the vector  $F$ .

Now what is the length of the vector  $F$ ? Well, we know what  $F$  is. It's  $x$  over  $\rho$  cubed  $i$  hat, plus  $y$  over  $\rho$  cubed  $j$  hat, plus  $z$  over  $\rho$  cubed  $k$  hat. So OK, so you compute the length of that vector, and what do you get? Well, it's exactly  $1$  over  $\rho$  squared. OK.

But we said that this is a sphere. I guess I didn't write it down. Let me write it down right here. This is a sphere whose radius is big  $R$ . It doesn't really matter very much what  $R$  we choose, we just want it to be big enough so that it contains the whole cube. If you said this a sphere of

radius 10, that would completely do the trick. That would be totally fine.

OK, so the radius of the sphere is big  $R$ , so the length of the field, we said, back over here, is  $1$  over  $R$  squared. The length of the vector  $F$ . So this flux integral then, is the integral over the sphere  $S_2$  of a constant. So it's the integral over the sphere of  $1$  over  $R$  squared  $dS$ . But when you integrate a constant over a surface, what you get is just that constant times the surface area.

Well, what's the surface area? This is a sphere. It's easy to understand its surface area. Its surface area is  $4\pi R$  squared. Right?

So this is equal to the surface area, so that's  $4\pi R$  squared, times whatever that constant was. So the constant was  $1$  over  $R$  squared. And so the  $R$  squareds cancel. Right? This is why it didn't matter what  $R$  we chose, because they're just going to cancel at the end, anyhow. OK, so those cancel, and we're left with  $4\pi$ .

So let's just quickly recap what we did in this part c. We're looking to compute the flux over the cube. But it's a kind of unpleasant integral we'd have to compute, to total up the fluxes over these various different faces and so on. So instead, we had this clever idea that we'll apply the divergence theorem to replace the cube with a more congenial surface. So the surface we choose, because this is a nice radial vector field-- that's our main hint. Because there was a  $\rho$  involved in the problem, if you will.

So the surface that we choose is some big sphere. And then we apply the extended Gauss's theorem to the solid region between the cube and the sphere. Outside the cube, but inside the sphere. So because the divergence of the field is  $0$ , the extended Gauss's theorem tells us that the two fluxes-- the flux out of the cube and the flux out of the sphere-- are actually equal to each other. But since the fluxes are actually equal to each other, in order to compute the flux out of the cube, it's enough to compute the flux out of the sphere. OK.

But computing the flux out of the sphere is relatively easy, because on the sphere, the integrand  $F \cdot n$  is just a constant. And so then we're integrating a constant over the surface of a sphere, and that just gives us the surface area of the sphere times that constant, which is  $4\pi R$  squared times  $1$  over  $R$  squared, which is  $4\pi$ . So the flux out of the cube then is also equal to  $4\pi$ . I'll stop there.