

18.02SC Multivariable Calculus, Fall 2010  
Transcript – Recitation 68, More Stokes' Theorem

Hi. Welcome back to recitation. In lecture you've been learning about vector calculus, Stokes' theorem, all sorts of cool stuff like that, curl. So I have a nice problem here for you. So let  $F$  be the following vector field. So it's the vector field whose direction is  $\hat{x} + \hat{y} + \hat{z}$ , but in addition I want to multiply it by  $\rho^n$ . So this is your usual  $\rho$  from spherical coordinates. This is the square root of  $x^2 + y^2 + z^2$ . And  $n$  can be any number. So it might be positive, it might be negative, it might be 0, doesn't have to be an integer. Some number, though, it's some constant.

So what I'd like you to do is to show that for this field, regardless of what the value of  $n$  happens to be, that it is a gradient field. So it is the gradient of some function. So why don't you pause the video, have a go at that, come back and we can work on it together.

So recall that for something to be a gradient field, what that means-- well, first of all, that means that there is some function that has it as the gradient, and we know a lot of other characterizations of it. And one of them that we know involves the curls. So let's talk about that one. So in order to look at the curl of this field, I'm going to want to take partial derivatives of its components. And so I'm going to want to take partial derivatives of  $\rho$ .

So let's just remember or recall that  $\frac{\partial \rho}{\partial x} = \frac{z}{\rho}$ . And similarly,  $\frac{\partial \rho}{\partial y} = \frac{y}{\rho}$ . And  $\frac{\partial \rho}{\partial z} = \frac{z}{\rho}$ . So you can just check that using the fact that you know what  $\rho$  is. Et cetera. I'm going to write et cetera because it's the same for the other two.

So let's call-- for shorthand-- let's call  $F = M\hat{i} + N\hat{j} + P\hat{k}$ . So  $M$ ,  $N$ , and  $P$  are its components. If  $F$  is this, then we know that  $\text{curl } F$  is equal to-- what is it? So it's  $\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}$ ,  $\hat{i}$ , plus-- OK, so it's the  $\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}$ ,  $\hat{j}$ , plus-- it's going to be, what's the last one?-- the  $\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$ ,  $\hat{k}$ . So if this is our formula for  $F$ , then this is our formula for the curl of  $F$ . And OK, so now we just have to compute these various different partial derivatives in order to see what the curl is. And then hopefully that'll tell us something about this field.

So let's see. So  $P_y$ . In our case, so  $M$  is equal to  $\rho$  to the little  $n$  times  $x$ . Big  $N$  is equal to  $\rho$  to the little  $n$  times  $y$ . And big  $P$  is equal to  $\rho$  to the little  $n$  times  $z$ . So let's look at our components here. So  $P$  sub  $y$ -- well,  $z$  is a constant with respect to  $y$ . So this is just  $\rho$  to the  $n$ . Well,  $z$  times the  $y$  partial of  $\rho$  to the  $n$ . So that's by the chain rule, so we got  $n \rho$  to the  $n$  minus  $1$  times  $y$  over  $\rho$  times  $z$ . So we can rewrite this as  $n y z \rho$  to the  $n$  minus  $2$ .

And similarly, so that was  $P$  sub  $y$ , so let's look at  $N$  sub  $z$ . So  $N$  is  $\rho$  to the  $n$  times  $y$ . So you take the  $z$  partial derivative, so  $y$  is a constant. So we need to look at the  $z$  partial derivative of  $P$  to the  $n$ . So this is, again, it's  $n$ -- sorry, I think I said  $P$  to the  $n$ . But of course this isn't a  $P$ , this is a  $\rho$ . So it's  $n \rho$  to the  $n$  minus  $1$  times partial  $\rho$  partial  $z$ . So that's  $z$  over  $\rho$  times  $y$ . And again, this is equal to  $n y z \rho$  to the  $n$  minus  $2$ .

OK, so  $P$  sub  $y$ , the  $y$  partial of  $P$  is  $n y z \rho$  to the  $n$  minus  $2$ . And the  $z$  partial of  $n$  is  $n y z \rho$  to the  $n$  minus  $2$ . And they're the same. So what does that mean? So that means the first component of curl of  $F$  is  $0$ . So if you do a little bit more arithmetic of exactly the same sort, you have two more components to check. What you're going to find is that the other ones are  $0$  also.

I'm not going to do out all those partial derivatives for you, but I trust that you can compute the similar looking partial derivatives that appear in these other two components-- this  $j$  should have had a hat-- the other partial derivatives that appear in these components and show that they're all also equal to  $0$ . So in our case, curl of  $F$  is just equal to  $0$  vector. OK, great.

So what does that mean? Well, we want to show that something is a gradient field. So we know that we can do that. We know that that happens precisely when its curl is  $0$  if it's defined on a simply connected domain. On a simply connected region. So we know that a function is a gradient field if it's defined on a simply connected domain and has curl  $0$ .

Well, let's see. So where is this thing defined? Where is  $F$  defined? Well, if  $n$  is bigger than or equal to  $0$ -- well, I guess I want strictly bigger than  $0$  just to be on the safe side. If  $n$  is positive, this is defined everywhere and we're great. If  $n$  is negative, then we have a problem at  $0$ . Because then we have division by  $\rho$ . So we don't want to divide by  $0$ . So if  $n$  is negative, there's a problem at the origin.

So in the worst case,  $F$  is defined everywhere except the origin. But that's simply connected. Because we're in space. If we were in the plane, this would be a different story. But in space, just removing a

point doesn't destroy simply connectedness. So this field  $F$  is defined everywhere, except the origin perhaps, so we're in a simply connected region. So when you have  $\text{curl } F$  in a simply connected region, your field is, in fact, a gradient field.

Now I leave it as an exercise to you to come up with the actual function, or one of the actual functions, of which this is a gradient. But we've just shown that because its curl is 0, and because it's defined in a simply connected region of space, that this field really is the gradient field of some function. So I'll stop there.

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