

JOEL LEWIS: Hi. Welcome back to recitation. In lecture, you've been learning about line integrals of vector fields. And I have one problem here that requires you to do two line integrals of vector fields on that subject. So OK.

So in both problems, F is going to be the vector field whose coordinates are x^2y and $x^2 + y^2$. And C is going to be the arc of the parabola $y = x^2$ that starts at the point $(1, 1)$ and ends at the point $(2, 4)$. All right? So what I'd like you to do is to compute the integral over this curve C of $F \cdot dr$ in two different ways.

So the first time, I'd like you to use the natural parametrization $x = t$, $y = t^2$. For me, at least, that's the first parametrization that I think of when I think about this curve. But I'd also like you to do it again using a different parametrization, using the parametrization $x = e^t$, $y = e^{2t}$. So this still parametrizes the curve $y = x^2$, since $e^{2t} = (e^t)^2$.

So why don't you pause the video, have a go at computing this integral using these two different parametrizations, come back, and we can work it out together.

So let's get started. We want to compute a line integral of a vector field F . So we know that the integral over our curve C of $F \cdot dr$ -- well, usually we write-- $F = M, N$. The two components are M and N . So in this case, this is going to be equal to the integral over C of $M dx + N dy$.

Now, in our case, we know what F is. We know that F is the vector field $x^2y, x^2 + y^2$. So this is equal to the integral over C of $x^2y dx + (x^2 + y^2) dy$. And now we just need to plug in in our two different cases, into our parametrizations, and this will turn into an integral that we can evaluate. So let's have a look-see.

So in our first parametrization over here, we have $x = t$ and $y = t^2$. And we want this parametrization to go from the points $(1, 1)$ to the point $(2, 4)$. So that means that t -- x -- t is going from 1 to 2. OK.

So I'm going to just write equals, and continue on. So in this first part-- so t is our parameter, and it's going from 1 to 2. So x^2t is going to be t times t^2 , and dx is dt . And then plus-- so $x^2 + y^2$ is $t^2 + t^4$. And dy is $2t dt$, since y is t^2 . All right. So this is our integral.

And now it's straightforward to compute. This is a single variable integral of a polynomial, so we might have some fraction arithmetic in our future, but nothing horrible.

So let's collect our dt's. So this is the integral from 1 to 2. So here I've got $2t^3 + 2t^5$ to the fifth. And here I have t^3 . So that's $3t^3 + 2t^5$ dt.

OK, and so now we have to evaluate. So this is equal to-- let me just continue it. So $3t^3$ becomes $3t^4$ over 4. Plus $2t^5$ becomes-- well, t^6 over 6-- $2t^6$ over 6, so that's plus t^6 over 3, between t equals 1 and 2. And let me just bring that computation up here.

So that's equal to 3 times 16 over 4, plus 64 over 3, minus $3/4$ plus $1/3$. OK, and now we gotta work this out. So this is maybe 12, and the thirds give me 21, so that's 33 minus $3/4$. So that's 32 and $1/4$, or 129 over 4, if you prefer. OK, so that's going to be our answer there.

So now let's go back and do it again using this alternate parametrization. So in this alternate parametrization, we had x equal e^t , and y equal e^{2t} . So in that case, that means that dx -- so x is e^t , y is e^{2t} , so our integral that we want, let me rewrite it. Well, actually, let me not rewrite it. Let me just go back here and take a look at it.

So we want the integral of $F \cdot dr$, and we know that that's equal to the integral over C of $x^2y dx + x^2 + y^2 dy$. So x is e^t , y is e^{2t} , so that means dx is going to be $e^t dt$ and dy is going to be $2e^{2t} dt$.

So OK. So we'll just have to plug these things in. So our integral-- maybe I'll give it a name: What We Want. WWW. What We Want is equal to the integral of, well we start plugging in. So $x^2y dx$ is e^t times e^{2t} times, we said dx is $e^t dt$. Plus, now $x^2 + y^2$ squared, so that's e^{2t} -- that's x^2 -- plus e^{4t} -- that's y^2 -- times $2e^{2t} dt$. That's dy . Whew. OK.

But this integral is missing something, right? Because it needs to be a definite integral. Right now, I've written down an indefinite integral. So that's a problem. So we need to think, what are the bounds on this curve? What are we integrating from and to? So let's go all the way back and look at the problem over here.

So this is the parametrization of the curve, and we need it to connect the points $(1, 1)$ and $(2, 4)$. So we need to know which value of t puts this parametrization at the point $(1, 1)$, and which

value of t puts it at the point $(2, 4)$.

Well, I think the easiest way to do that is to solve this, and we have t equals natural log of x . So natural log of 1 is 0, and natural log of 2 is natural log of 2. So OK. So we want t to go from 0 to natural log of 2. To go from this point to that one.

OK, so 0 to natural log of 2. Let's remember that, and come back over here and put that in. So we want this integral that we're trying to compute using this alternate parametrization-- we've already written down what the integrand is-- but it's going to be the integral, we just said, from 0 to the natural log of 2. OK. Great. So now let's take this integral and let's start simplifying it.

So let's see. This is an e to the t times e to the $2t$ times e to the t , so that's e to the $4t$ dt . And here we have-- that looks like a $2e$ to the $4t$ plus $2e$ to the $6t$ dt . So if we combine like terms, we get the integral from 0 to natural log of 2. So we had an e to the $4t$ and a $2e$ to the $4t$, so that's $3e$ to the $4t$ plus $2e$ to the $6t$ dt .

OK, $3e$ to the $4t$ plus $2e$ to the $6t$. OK. And this is also a different-looking integral than the one we had before, but it's not a difficult one to evaluate, because it's just e to a constant times t dt . Pretty straightforward stuff, I think. So what does it actually give us? Let me come up here and write it up here.

So this is equal to-- all right, integrate e to the $4t$, and you get e to the $4t$ over 4, so this is $3/4 e$ to the $4t$, plus $2/6$ -- so that's plus $1/3 e$ to the $6t$, and we're evaluating that between 0 and $\ln 2$. And I will leave it to you to confirm that what you get when you do this is you get exactly 129 over 4 when you plug in these values and take the difference.

So one thing to notice here is that doing it with the second parametrization, we got 129 over 4. Now, if you look back just right here, you see that when we did it with the first parametrization, we also got 129 over 4. Now this is a general phenomenon, right? The parametrization doesn't change the value of the line integral.

So when you have a line integral of a vector field, you can parametrize the curve that you're integrating over in a bunch of different ways. I mean, in fact, infinitely many different ways. There are all sorts of different choices, and they all give the same answer. They have to. So no matter what parametrization you choose, you get the same number out for the line integral over the same curve of the same vector field. All right.

So what that means for you is that when you're given just a curve and a vector field and you get to choose your parametrization, you should always just choose whatever the nicest one is. Because it doesn't matter which one you choose, and there's no reason ever to choose anything harder than necessary. All right.

So now, I want to quickly recap what we did. So we had a curve and a vector field F , and we were asked to compute the line integral of $F \cdot dr$ and we did it two different ways.

So in both cases, we used the fact that $F \cdot dr$ is $M dx$ plus $N dy$. We wrote that out using the known values of M and N , and using the known components of F . And then we plugged in for both cases.

In this case, we used that first parametrization. And then over to my left, we used this second parametrization that I gave you. And in both cases, that reduced it to an integral in terms of a single variable t that was the parameter that we could compute. And the answer came out the same both times, as it had to.

So one thing you can notice is if you compare this with the preceding lecture video, you'll see that this curve and vector field, they connect the same points and it's the same vector field, and these answers both differ from both of the answers in the preceding recitation video. So this is further illustration that different curves can give you different values, but if you take the same curve, different parametrizations always have to give you same values for that line integral of the same vector field. I'll end there.