V15.2-3 Relation to Physics

The three theorems we have studied: the divergence theorem and Stokes' theorem in space, and Green's theorem in the plane (which is really just a special case of Stokes' theorem) are widely used in physics and continuum mechanics, in the study of fields, potentials, heat flow, wave motion in liquids, gases, and solids, and thermodynamics, to name some of the uses. Often partial differential equations which model some physical situation are derived using the vector integral calculus theorems. This section is devoted to a brief account of where you will first meet the theorems: in electromagnetic theory.

2. Application to Maxwell's equations.

Each of Maxwell's equations in electromagnetic theory can be written in two equivalent forms: a differential form which involves only partial derivatives, and an integrated form involving line, surface, and other multiple integrals.

In a sense we have already seen this with our criterion for conservative fields; we assume $\mathbf{F}$ is continuously differentiable in all of 3-space. Then the integrated form of the criterion is on the left, and the differential form is on the right:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = 0 \quad \text{for all closed } C \quad \Leftrightarrow \quad \text{curl } \mathbf{F} = \mathbf{0} \quad \text{for all } x, y, z \ .$$

And we know that it is Stokes' theorem which provides the bridge between these two equivalent forms of the criterion.

The situation with respect to Maxwell's equations is similar. We consider here two of them, as typical.

Gauss-Coulomb Law. Let $\mathbf{E}$ be an electrostatic field, arising from a distribution in space of positive and negative electric charge. Then the Gauss-Coulomb Law may be written in either of the two forms

$$(8) \qquad \nabla \cdot \mathbf{E} \ = \ 4\pi\rho, \qquad \rho \ = \ \text{charge density;} \qquad \textit{(differential form)}$$

$$(8') \qquad \iint_S \mathbf{E} \cdot d\mathbf{S} \ = \ 4\pi Q, \qquad Q \ = \ \text{total net charge inside } S. \qquad \textit{(integrated form)}$$

These are two equivalent statements of the same physical law. The integrated form is perhaps a little easier to understand, since the left hand side is the flux of $\mathbf{E}$ through S, which is a more intuitive idea than div $\mathbf{E}$. On the other hand, quite a lot of technique is required actually to calculate the flux, whereas very little is needed to calculate the divergence.

Neither (8) nor (8′) is mathematics — both are empirically established laws of physics. But their *equivalence* is a purely mathematical statement that can be proved by using the divergence theorem.

Proof that (8) $\Rightarrow$ (8′)**.**

Let D be the interior of the closed surface S. Then

$$\begin{aligned} \iint_S \mathbf{E} \cdot d\mathbf{S} \ &= \ \iiint_D \nabla \cdot \mathbf{E} \ dV \qquad \text{by the divergence theorem;} \\ &= \ 4\pi \iiint_D \rho \ dV \qquad \text{by (8)} \\ &= \ 4\pi Q \ , \qquad \text{by definition of } \rho \text{ and } Q. \end{aligned}$$

Proof that $(8')\ \Rightarrow\ (8)$**.**

We reason by contraposition: that is, we show that if (8) is false, then $(8')$ must also be false.

If (8) is false, this means that we can find some point $P_0 : (x_0, y_0, z_0)$ where $\mathbf{E}$ is defined and such that $\nabla \cdot \mathbf{E} \ \neq\ 4\pi\rho$ at P_0; we write this inequality as

$$\nabla \cdot \mathbf{E} - 4\pi\rho \ \neq\ 0, \quad \text{at } P_0.$$

Say the quantity on the left is positive at P_0. Then by continuity, it is also positive in the interior of a small sphere S_0 centered at P_0; call this interior B_0. Then

$$\iiint_{B_0} (\nabla \cdot \mathbf{E} - 4\pi\rho)\, dV \ >\ 0\ ,$$

which we write

$$\iiint_{B_0} \nabla \cdot \mathbf{E}\, dV \ > 4\pi \iiint_{B_0} \rho\, dV.$$

The integral on the right gives the total net charge Q_0 inside S_0; applying the divergence theorem to the integral on the left, we get

$$\iint_{S_0} \mathbf{E} \cdot d\mathbf{S} \ >\ 4\pi Q_0$$

which shows that $(8')$ is also false, since it fails for S_0. □

Faraday's Law A changing magnetic field $\mathbf{B}(x, y, z; t)$ produces an electric field $\mathbf{E}$. The relation between the two fields is given by Faraday's law, which can be stated (in suitable units) in two equivalent forms (c is the velocity of light):

$$\nabla \times \mathbf{E} \ =\ -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} \qquad \textit{differential form} \tag{9}$$

$$\oint_C \mathbf{E} \cdot d\mathbf{r} \ =\ -\frac{1}{c}\frac{d}{dt}\iint_S \mathbf{B} \cdot d\mathbf{S} \qquad \textit{integrated form} \tag{9'}$$

As before, it is the integrated form which is more intuitive, though harder to calculate. The line integral on the left is called the *electromotive force* around the closed loop C; Faraday's law $(9')$ relates it to the magnetic flux through any surface S spanning the loop C.

A few comments on the two forms. The derivative in (9) is taken by just differentiating each component of $\mathbf{B}$ with respect to the time t. It is a partial derivative, since the components of $\mathbf{B}$ are also functions of x, y, z. In $(9')$ on the other hand we have an ordinary derivative, since after the integration, the flux is a function of t alone.

It is understood in physics that on S the positive direction for flux and the positive direction on C must be compatibly chosen.

The magnetic flux through S is the same for all surfaces S spanning the loop C. (This is a consequence of the physical law $\nabla \cdot \mathbf{B} = 0$.) As a result, one speaks simply of "the flux through the loop C", meaning the flux through any surface spanning C, i.e. having C as its boundary.

Once again, though (9) and (9′) both express the same physical law, the equivalence between them is a mathematical statement; to prove it we use Stokes' theorem.

Proof that (9) $\Rightarrow$ (9′)**,**

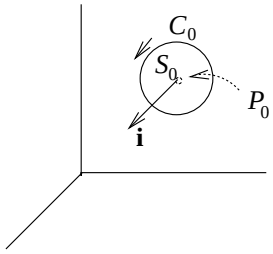
$$\begin{aligned} \oint_C \mathbf{E} \cdot d\mathbf{r} &= \iint_S \nabla \times \mathbf{E} \cdot d\mathbf{S}, && \text{by Stokes' theorem,} \\ &= -\frac{1}{c} \iint_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S}, && \text{by (9)} \\ &= -\frac{1}{c} \frac{d}{dt} \iint_S \mathbf{B} \cdot d\mathbf{S}\ , \end{aligned}$$

if $\mathbf{B}$ has a continuous derivative and S is smooth, and finite in extent and in area. (This last equality is fairly subtle, and is taken up in theoretical advanced calculus courses.) □

Proof that (9′) $\Rightarrow$ (9). We show that if (9) is false, then (9′) is false:

If (9) is false, this means that at some point P_0, $\nabla \times \mathbf{E} \neq -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t}$; we write this

$$\nabla \times \mathbf{E} + \frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} \neq \mathbf{0}. \tag{10}$$



This means that at least one of the components of this vector is not 0 at P_0; say it is the $\mathbf{i}$-component, and it's positive. Then by continuity it will remain positive in a small ball around P_0. Inside this little ball, draw a little disc S_0 as shown with center at P_0, having normal vector $\mathbf{i}$; orient its circular boundary C_0 compatibly.

Since the vector on the left in (10) has a positive $\mathbf{i}$-component on S_0,

$$\iint_{S_0} \left(\nabla \times \mathbf{E} + \frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} \right) \cdot d\mathbf{S} > 0,$$

which we may write

$$\iint_{S_0} \nabla \times \mathbf{E} \cdot d\mathbf{S} > -\frac{1}{c} \iint_{S_0} \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S};$$

applying Stokes' theorem to the left-hand side, and interchanging the order of differentiation and integration on the right (this is valid under the reasonable hypotheses we stated before), we get

$$\oint_C \mathbf{E} \cdot d\mathbf{r} > -\frac{1}{c} \frac{d}{dt} \iint_S \mathbf{B} \cdot d\mathbf{S},$$

which shows the integrated form (9′) is false for this little circle and disc, and therefore not true in general. □

3. Harmonic functions in space.

A harmonic function in space is by definition a function $f(x, y, z)$ which satisfies Laplace's equation $\nabla^2 f = 0$, or written out (see (5)):

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \;=\; 0\ .$$

For example, the potential function for an electrostatic field $\mathbf{E}$ is harmonic in any region of space which is free of electrostatic charge. Similarly, the potential function for a gravitational field $\mathbf{F}$ is harmonic in any region where there is no mass. These statements are mathematical consequences of physical laws, and therefore are also physical laws — i.e., experimental facts, not mathematical facts.

To see why the potential function for $\mathbf{E}$ is harmonic, suppose we are in a simply-connected region of space where there is no charge. We then have

$$\begin{aligned} \textit{Gauss-Coulomb law} \qquad & \nabla \cdot \mathbf{E} \;=\; 0 \qquad \text{since } \rho = 0 \text{ in the region} \\ \textit{Faraday's law} \qquad & \nabla \times \mathbf{E} \;=\; 0; \end{aligned}$$

the second equation is valid since the field arises from a distribution of static electric charges — there is no changing magnetic field. Faraday's law shows that $\mathbf{E}$ is conservative, so that it has a (mathematical) potential function $f(x, y, z)$; the physical potential function would be $-f(x, y, z)$. By the Gauss-Coulomb law, noting that $\mathbf{E} = \nabla f$, we get

$$\nabla \cdot \mathbf{E} = \nabla \cdot \nabla f = 0, \qquad \text{or} \qquad \nabla^2 f \;=\; 0,$$

showing that $f(x, y, z)$ is a harmonic function.

Because harmonic functions can represent potential functions, there is great interest in finding harmonic functions in a region D of space. Typically, one prescribes the values that $f(x, y, z)$ should have on the boundary of D, and then searches analytically (or numerically by computer) for the values of $f(x, y, z)$ inside D. In this work, the divergence theorem gives an important theoretical tool; some of the Exercises use it to explore the situation a little further.

In general, this aspect of the subject properly belongs to the realm of partial differential equations, i.e., to Differential Equations and Advanced Calculus courses: see you there, maybe.

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