

**CHRISTINE  
BREINER:**

Welcome back to recitation. In this video, I'd like to do two problems that ask us to determine the flux of a vector field along a surface.

So the first one is I'd like you to find the outward flux of the vector  $z \mathbf{i} + x \mathbf{j} + y \mathbf{k}$  through the piece of the cylinder that's shown. So it's just shading the cylindrical part. So it's a cylinder of radius  $a$ , and you're taking the piece in the first octant up to height  $h$ . So that's the first question I'd like you to answer, is the flux of this vector field through that piece of the cylinder, the outward flux.

And then I'd also like you to find the outward flux of this vector field  $x^2z \mathbf{i} + y^2z \mathbf{j} + z^3 \mathbf{k}$  through the piece of the sphere of radius  $a$  in the first octant. So again, it will be in the first octant, like this one is in the first octant. It'll be the piece of the sphere that sits in the same part of three-dimensional space as the piece of the cylinder we're looking at here.

So what I'd like you to do, again, is just find the flux-- in both cases, the outward flux-- of the vector listed through the surface that is listed. And then when you feel comfortable and confident with your answer, you can bring the video back up and I'll show you how I did it.

OK, welcome back. So again, what we're trying to do is determine flux of a vector field through a surface. So I am going to determine first what the normal is here, and then what  $\mathbf{F} \cdot \mathbf{n}$  is here, and then using the fact that I know  $\mathbf{F} \cdot \mathbf{n}$  and I know  $dS$  in a good parametrization, I'm going to be able to calculate the integral quite simply.

So let's point out first, that the normal at any point on the surface is going to be equal to  $x \mathbf{i} + y \mathbf{j} + 0 \mathbf{k}$  divided by  $a$ , where  $a$  is, again, the radius. Because in the normal, I know there's no  $z$ -component. And I actually know it's the same as the normal would be on a circle of radius  $a$ .

And so this vector  $x \mathbf{i} + y \mathbf{j}$  has length  $a$  obviously, so that's why I'm dividing by  $a$ . So when I look at the normal and I dot it with  $\mathbf{F}$ , let's see what I get.  $\mathbf{F}$  dotted with the normal is going to be  $x^2z + x^2y$  divided by  $a$ , right? So I just took  $x$  dotted with  $x$  times  $z$ ,  $y$  times  $z$ , and  $0$  times  $y$ , I add those up, and I still have to divide by my  $a$ . So that's actually the vector dotted with the normal.

Now, what's a natural parametrization to use here is obviously the cylindrical coordinates, because I'm on a cylinder. The radius is fixed. But what I'm interested in is changes in  $\theta$

and changes in the height. So I'm going to be interested in  $d\theta$  and  $dz$ , and I need to understand what  $dS$  in this case is. And it's just a  $d\theta dz$ .

So let me write down what I'm going to need to completely determine the rest of number one is I'm integrating over the surface. And I'll put the bounds momentarily. Actually, I should also point out, I'm going to write  $x$  and  $y$  in terms of  $\theta$ , because now I know what they are.  $x$  in terms of  $\theta$  is a  $\cos\theta$ , and  $y$  in terms of  $\theta$  is a  $\sin\theta$ .

So when I simplify the expression  $F \cdot n$ , I get  $z$  times  $\cos\theta$  for my first component. And I get  $a$  times  $\cos\theta \sin\theta$  for my second component. So let me maybe point out again how I got that. Let me come over here.

$x$  is a  $\cos\theta$ . So I get  $z$  times a  $\cos\theta$  divided by  $a$ . So I get  $z$  times  $\cos\theta$ .  $x$  again is a  $\cos\theta$ .  $y$  is a  $\sin\theta$ . So I get a squared  $\cos\theta \sin\theta$ . And I divide by  $a$ , so I get a single  $a \cos\theta \sin\theta$ .

And then my  $dS$ , as I mentioned before, is a  $d\theta dz$ , right?

And so now I just have to figure out the bounds in  $\theta$  and the bounds in  $z$ . Well, the bounds in  $z$  are very easy. The bounds in  $z$  are simply 0 to  $h$ , and so I'm going to put those here on the outside.

And then the bounds in  $\theta$ . Maybe it's helpful to come over and look at my picture. This direction, in the  $x$ -direction, is  $\theta = 0$ . When I swing around to the  $y$ -direction, I'm at  $\theta = \pi/2$ . So I need to go from 0 to  $\pi/2$ , right? So let me come back over here.

OK, so now, really, I have a couple of constants that are letters, but everything now, I'm ready to integrate. So  $a$  is just a constant, and  $h$  is a constant, and  $z$  and  $\theta$  are my variables, and I can actually do this integration quite simply. I'm not going to do it because I know this is at this point something we already know how to do, but I'll give you the answer. So let me write down the answer.

You get a  $h^2/2$ , plus a squared  $h/2$ . So that's the solution you actually get.

So again, I mean, you have to integrate in  $d\theta$  first. So you have to deal with this, you have to deal with this, and this uses a good trig identity. You can use the fact that  $2 \sin\theta \cos\theta$  is  $\sin 2\theta$ . I'll give you that little hint, and then you can figure it out from there.

And then you integrate in  $z$ , and you evaluate from 0 to  $h$ . So that's the solution to number one.

So now, let's look at number two. In number two, we have to again figure out-- we have our vector field and we know our surface is a piece of a sphere. And so if we're going to parametrize the surface of a sphere, we know we want to use  $d\theta$  and  $d\phi$ , OK?

And let me point out first, that again, the normal is going to be in some ways similar to what we saw on the cylinder, but now instead of  $x$  comma  $y$  comma 0, because it's a sphere, it's going to be  $x$  comma  $y$  comma  $z$  divided by  $a$ . So again, as before, let me just point out the normal that I'm going to be using is  $x$  comma  $y$  comma  $z$ , all divided by  $a$ .

So I'm going to dot  $F$  with  $n$  and look at the surface integral with respect to  $dS$ --  $d$  capital  $S$  there-- and I'll see what I get. So again, I know how I'm going to parametrize this sphere. I already mentioned it, but let me say it again. It's going to be in  $\theta$  and  $\phi$ , right? Because we have a constant radius. We're on a sphere of radius  $a$ . So I don't need to change  $\rho$ . It's a two-dimensional thing. So  $\theta$  is varying and  $\phi$  is varying.

So let's see what we get first. Let me do a little work and see what we get when we look at  $F$  dotted with  $n$ . So let me first point out that  $F$  dotted with  $n$  looks like it's  $x^2 z$  plus  $y^2 z$  plus-- just to make this obvious--  $z^3$ , all divided by  $a$ .  $x^2$  plus  $y^2$  plus  $z^2$  is  $a^2$ , right? So it's actually  $a^2 z$  divided by  $a$ . So it's just  $a z$ , right?

So far, all I've done was I dotted  $F$  with the normal, and I knew the fact that  $x^2$  plus  $y^2$  plus  $z^2$  was  $a^2$ , because I was on a sphere of radius  $a$ . So  $a^2 z$  divided by  $a$  is  $a z$ . And now, if I want to use the right coordinates, think about  $\theta$  and  $\phi$ .  $z$  is  $a \cos \phi$ . So  $F \cdot n$  in the coordinates I'm interested in is going to be  $a^2 \cos \phi$ . So let me get out of the way so you can see that. So that's what our  $F \cdot n$  will be.

And now, if we're going to figure out the flux, of course, it's the integral of  $F \cdot n \, dS$ . And let's remind ourselves what  $dS$  is.  $dS$  is going to be  $a^2 \sin \phi \, d\theta \, d\phi$ . You saw this in lecture, actually, also. So this should look familiar.

And so now I just have to integrate  $F \cdot n \, dS$  over the right bounds for  $\theta$  and  $\phi$ . So let's determine what those are. I'll put everything together, and we'll determine what those are.

So I've got  $F \cdot n \, dS$ . That's going to be  $a \cos^4 \theta \sin \theta \, d\theta \, d\phi$ . And let's think about what is the picture that I need in terms of the first octant of a sphere. Maybe I should draw a quick picture over here so we can remember what that looks like.

So it's going to be-- this is not going to be the greatest drawing ever-- but it's something like this. And so I've got pieces of a circle at each level. I've got a piece of a circle, right? If this is the x-direction, this is the y-direction, and this is the z-direction,  $\theta$  is going from 0-- again-- to  $\pi/2$ , and  $\phi$  is going from 0 to  $\pi/2$ , right? So they're both going from 0 to  $\pi/2$ .

So hopefully, you were able to get this far at least, in terms of figuring out the flux of  $F$  through that piece of the sphere. And I'm, again, just going to write down the solution, and then you can check your answer against the solution. And I got  $a^4/4\pi$ . So this whole solution is  $a^4/4\pi$ . So you can check your solution there.

Again, I want to point out, what we did in both of these problems is we were trying to compute the flux of a certain vector field through a surface. And if you'll notice, what I actually did in this case is I kept the parametrization in terms of the  $x, y, z$  variables first, and then I put it in the parametrization of  $\theta$  and  $\phi$  after. And that made it a little easier to hang on to and figure out what it was. Because notice, that  $x, y,$  and  $z$  become very complicated in  $\theta$  and  $\phi$ . I have to write a lot more down, I guess. And then simplify things more carefully.

This way, it was very obvious that I got an  $a^2 z$  in the numerator. So sometimes it's a little easier to compute the  $F \cdot n$  in the initial  $x, y, z$  variables, and then change it to the appropriate parametrization for the surface you're looking at.

So that's what we did, actually. In both cases we computed  $F \cdot n$ , we put it in the right parametrization, and then we had to figure out what  $dS$  was. We had to make sure we knew  $dS$ , and then we just had to integrate over the appropriate bounds for our parameters. And that's giving us the flux across the surface.

So I think that's where I'll stop.