18.02   Final Exam

No books, notes or calculators.
15 problems, 250 points.

Useful formula: $\cos^2(\theta) = \frac{1}{2}(1 + \cos(2\theta))$

Problem 1. (20 points)
a) (15 pts.) Find the equation in the form $Ax + By + Cz = D$ of the plane $\mathcal{P}$ which contains the line L given by $x = 1 - t, \quad y = 1 + 2t, \quad z = 2 - 3t$ and the point (- 1, 1, 2).

b) (5 pts.) Let $\mathcal{Q}$ be the plane $2x + y + z = 4$. Find the component of a unit normal vector for $\mathcal{Q}$ projected on a unit direction vector for the line L of part(a).

Problem 2. (15 points)
Let L denote the line which passes through (0,0,1) and is parallel to the line in the xy-plane given by $y = 2x$.

a) (5 pts.) Sketch L and give its equation in vector-parametric form.

b) (5 pts.) Let $\mathcal{P}$ be the plane which passes through (0,0,1) and is *perpendicular* to the line L of part(a). Sketch in $\mathcal{P}$ (above) and give its equation in point-normal form.

c) (5 pts.) Suppose that the point $P \neq (0, 0, 1)$ lies on L. Write down the method or formula you would use to find the point P^* which is: (i) on L; (ii) the same distance away from the point (0,0,1) as P; and is (iii) on the *other* side of $\mathcal{P}$ from P.

Problem 3. (20 points)
Given the 3×3 matrix: $A_a = \begin{bmatrix} 1 & 0 & 3 \\ -2 & 1 & -1 \\ -1 & 1 & a \end{bmatrix}$:

a) (5 pts.) Let $a = 2$: show that $\mid A_2 \mid = 0$

b) (7 pts.) Find the line of solutions to $A_2 \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

c) (8 pts.) Suppose now that $a = 1$, and that $A_1^{-1} = \begin{bmatrix} * & * & * \\ -3 & p & 5 \\ * & * & * \end{bmatrix}$.   Find p.

Problem 4. (10 points)
Let $\mathbf{r}(t) = \langle \cos(e^t), \sin(e^t), e^t \rangle$.

a) (5 pts.) Compute and simplify the unit tangent vector $\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{\mid \mathbf{r}'(t) \mid}$.

b) (5 pts.) Compute $\mathbf{T}'(t)$

Problem 5. (20 points)

Consider the function $F(x, y, z) = z\sqrt{x^2 + y} + 2\dfrac{y}{z}$:

a) (10 pts.) The point $P_0 : (1, 3, 2)$ lies on the surface $F(x, y, z) = 7$. Find the equation of the tangent plane to the surface $F = 7$ at P_0.

b) (5 pts.) If starting at P_0 a small change were to be made in only *one* of the variables, which one would produce the largest change (in absolute value) in F ? If the change is this variable was of size 0.1, approximately how large would the change in F be ?

c) (5 pts.) What distance from P_0 in the direction $\pm\langle -2, 2, -1\rangle$ will produce an approximate change in F of size 0.1 units, according to the (already computed) linearization of F?

Problem 6. (15 points)

Let $f(x, y) = x + 4y + \dfrac{2}{xy}$.

a) (10 pts.) Find the critical point(s) of $f(x, y)$

b) (5 pts.) Use the second-derivative test to test the critical point(s) found in part(a).

Problem 7. (10 points)

Let $\mathcal{P}$ be the plane with equation $Ax + By + Cz = D$ and $P_0 = (x_0, y_0, z_0)$ be a point which is *not* on $\mathcal{P}$.

Use the Lagrange multiplier method to *set up* the equations satisfied by the point (x, y, z) on $\mathcal{P}$ which is *closest* to P_0. (Do not solve.)

Problem 8. (15 points)

a) (10 pts.) Let $F(x, y, z)$ be a smooth function of three variables for which $\nabla F(1, -1, \sqrt{2}) = \langle 1, 2, -2\rangle$.

Use the Chain Rule to evaluate $\dfrac{\partial F}{\partial \phi}$ at $(\rho, \phi, \theta) = (2, \frac{\pi}{4}, -\frac{\pi}{4})$.

(Use $x = \rho \sin\phi \cos\theta$, $y = \rho \sin\phi \sin\theta$, $z = \rho \cos\phi$.)

b) (5 pts.) Suppose $f(x, y)$ is a smooth, non-constant function. Is it possible that, for *all* points (x, y), the gradient of f at the point (x, y) is equal to the vector $\langle -y, x\rangle$? Justify (briefly).

Problem 9. (10 points)

$$\iint_R f\, dA = \int_0^2 \int_{x^2}^{2\sqrt{2x}} f(x, y)\, dy\, dx .$$

a) (5 pts.) Sketch the region R.

b) (5 pts.) Rewrite the double integral as an iterated integral with the order interchanged.

Problem 10. (15 points)

Set up the integral $\displaystyle\iint_R f(x,y)\,dA$ where R is the region bounded by the four curves

$x^2 y = 4$, $x^2 y = 9$, $\frac{y}{x} = 1$, and $\frac{y}{x} = 2$ as a double integral in the variables $u = x^2 y$ and $v = \frac{y}{x}$. (Your answer should be completely ready to integrate, once the function f is given.)

Note: the inverse transformation is given by $x = u^{\frac{1}{3}} v^{-\frac{1}{3}}$, $y = u^{\frac{1}{3}} v^{\frac{2}{3}}$.

Problem 11. (15 points)

$\mathbf{F}(x,y) = x\,(\mathbf{i} + \mathbf{j})$, and let C be the closed curve in the xy-plane formed by the triangle with vertices at the origin and the points $(1,0)$ and $(0,1)$.

a) (5 pts.) Give a rough sketch of the field $\mathbf{F}$ in the first quadrant, and use it to predict whether the net flux out of the region R = the interior of C will be positive or negative.

b) (5 pts.) Compute the flux integral $\displaystyle\oint_C \mathbf{F} \cdot \hat{\mathbf{n}}\,ds$ directly.
(Specify which orientation you are using for C.)

c) (5 pts.) Compute the flux integral $\displaystyle\oint_C \mathbf{F} \cdot \hat{\mathbf{n}}\,ds$ using the appropriate *double* integral. (Set up, then using short-cut is ok.)

Problem 12. (20 points)

Let G be the solid 3-D cone bounded by the lateral surface given by $z = 2\sqrt{x^2 + y^2}$ and by the plane $z = 2$. The problem is to compute
$\bar{z}$ = the z-coordinate of the center of mass of G, in the case where the density is equal to the height above the xy-plane.

a) (5 pts.) Find the mass of G using cylindrical coordinates

(b) (5 pts.) Set up the calculation for $\bar{z}$ using cylindrical coordinates
(Answers should be ready to integrate out – but *do not evaluate.*)

(c) (10 pts.) Set up the calculation for $\bar{z}$ using spherical coordinates.
(Answers should be ready to integrate out – but *do not evaluate.*)

Problem 13. (15 points)

$\mathbf{F}(x,y,z) = \left(y + y^2 z\right)\mathbf{i} + \left(x - z + 2xyz\right)\mathbf{j} + \left(-y + xy^2\right)\mathbf{k}$

a) (3 pts.) Show that $\mathbf{F}(x,y,z)$ is a gradient field using the derivative conditions.

b) (10 pts.) Find a potential function $f(x,y,z)$ for $\mathbf{F}(x,y,z)$, using any *systematic* method. Show the method used and all work clearly.

c) (2 pts.) Find $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is the straight line joining the points $(2, 2, 1)$ and $(1, -1, 2)$ (in that order), using as little computation as possible.

Problem 14. (25 points)

In this problem S is the surface given by the quarter of the right-circular cylinder centered on the z-axis, of radius 2 and height 4, which lies in the first octant. The field $\mathbf{F}(x, y, z) = x\,\mathbf{i}$.

a) (5 pts.) Sketch the surface S and the field $\mathbf{F}$.

(Suggestion: use a coordinate system with y pointing out of the paper.)

b) (10 pts.) Compute the flux integral $\displaystyle\iint_S \mathbf{F} \cdot \hat{\mathbf{n}}\, dS$.

(Use the normal which points 'outward' from S, i.e. on the side away from the z-axis.)

c) (5 pts.) G be the 3D solid in the first octant given by the interior of the quarter cylinder defined above. Use the divergence theorem to compute the flux of the field $\mathbf{F} = x\,\mathbf{i}$ out of the region G.

d) (5 pts.) The boundary surface of G is comprised of S together with four other faces. What is the flux outward through these four faces, and why ? Use the answers to parts(b) and (c), and also verify using the sketch of part(a).

Problem 15. (25 points)

$\mathbf{F}(x, y, z) = (yz)\,\mathbf{i} + (-xz)\,\mathbf{j} + \mathbf{k}$. Let S be the portion of surface of the paraboloid $z = 4 - x^2 - y^2$ which lies above the first octant; and let C be the closed curve $C = C_1 + C_2 + C_3$, where the curves C_1, C_2 and C_3 are the three curves formed by intersecting S with the xy, yz and xz planes respectively (so that C is the boundary of S). Orient C so that it is traversed CCW when seen from above in the first octant.

a) (15 pts.) Use Stokes' Theorem to compute the loop integral $\displaystyle\oint_C \mathbf{F} \cdot d\mathbf{r}$ by using the *surface integral* over the capping surface S.

b) (10 pts.) Set up and evaluate the loop integral $\displaystyle\oint_C \mathbf{F} \cdot d\mathbf{r}$ directly by parametrizing each piece of the curve C and then adding up the three line integrals.

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18.02SC Multivariable Calculus
Fall 2010

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