

**DAVID JORDAN:** Hello, and welcome back to recitation. Today what we want to work on is drawing level curves. This is for all the artists out there in the audience. We have three functions here:  $z$  is  $2x$  plus  $y$ ,  $z$  is  $x$  squared plus  $y$  squared, and  $z$  is  $x$  squared minus  $y$  squared, and we want to get some practice drawing their level curves. Now, just to remind you, the level curves are not drawn in three dimensions. They're drawn in the  $xy$ -plane and they're constructed by setting  $z$  to be a constant and then graphing the curve that we get, so we can think about a relief map that we might use if we were hiking. So why don't you get started on that. Pause the video, and we'll check back, and I'll show you how I solve this.

Welcome back. So over here, we've got the equation for part a already set up. So  $z$  is  $2x$  plus  $y$ . So now, what we need to do to get started is just draw the  $xy$ -axis. And, you know, there's really not a precise science for drawing these level curves out. We just need to choose some values of  $z$  that we feel are representative and then just draw them in.

So one thing we notice about this is that if we choose  $z$  to be a constant, then the equations that we're going to get is  $2x$  plus  $y$  equals some constant, right? So, you know, these are just going to be lines. The level curves in this case are just going to be lines. So, for instance, if we take the level curve at  $z$  equals  $0$ , then we have just the equation  $2x$  plus  $y$  equals  $0$ . And so that has intercept-- so we're looking at-- so  $0$  equals  $2x$  plus  $y$ , so that's just  $y$  equals minus  $2x$ . So that's this level curve. That's the level curve at  $z$  equals  $0$ .

Now, if you think about it, all the other level curves, we're just going to be varying the constant here, and so we're just going to be shifting this line. So all of our level curves in this case are just straight lines. So let's see if we can make some sense out of that by thinking about the graph in three dimensions of this function.

So over here, I'm going to draw-- So this function  $z$  equals  $2x$  plus  $y$ , if we draw its graph, it's just a plane, right? So it's just a plane, which I'll just kind of draw in cartoon form, something like that. And now when we do level curves, what we're doing is we're slicing this plane with another plane, which is the horizontal values where  $z$  is a constant. And so, for instance, if we take the level curve here, then we're just intersecting these two planes, and their intersection is just a line, and that's exactly the lines that we're drawing here. So it's not surprising that we were graphing a linear function and that our contour lines, our level curves, were just straight lines.

So let's go on to a slightly more interesting example, which is part b, which I have written up here. So this is the function  $z$  equals  $x$  squared plus  $y$  squared. Actually, this is even easier to get started drawing the level curves for. Well, if you think about it, if I fix the value of  $z$ , then this is exactly the equation for the circle with radius square root of  $z$ . So level curves, level curves for the function  $z$  equals  $x$  squared plus  $y$  squared, these are just circles in the  $xy$ -plane. And if we're being careful and if we take the convention that our level curves are evenly spaced in the  $z$ -plane, then these are going to get closer and closer together, and we'll see in a minute where that's coming from.

So let's draw what's going on in three dimensions. So if we graph  $z$  equals  $x$  squared plus  $y$  squared in three dimensions, this is just a paraboloid opening up. And now what you can see is that if we slice this through the constant-- through  $z$  equals constant planes, then we're just getting these circles, and those are precisely the circles that we're drawing on the level curve. And because the parabola gets steeper and steeper, that's telling us that these circles, if we keep incrementing  $z$  in a constant way, that's telling us that the circles, which are the shadows below here, are going to get closer and closer and closer. This reflects the fact that this is getting steeper and steeper. In fact, this is generally true. If you're looking at a contour plot where the intervals between level curves are at regular distances, then very close contour lines means that the function is very steep there. So that's something to keep in mind.

Let's look at one more example. This is  $z$  equals  $x$  squared minus  $y$  squared. So to get started with this, well, again, if we start choosing constant values of  $z$ , this is just giving us hyperbolas, hyperbolas of two sheets. So, for instance, if we take-- so let's see what happens if we take  $z$  equals 0. So if we take  $z$  equals 0, then something a little special happens. This becomes  $x$  plus  $y$  times  $x$  minus  $y$  equals 0. We can factorize  $x$  squared minus  $y$  squared as  $x$  plus  $y$  times  $x$  minus  $y$ , and if this is 0, then that means either  $x$  plus  $y$  is 0 or  $x$  minus  $y$  is 0. So that tells us that the zero level curves for this graph are the lines  $y$  equals minus  $x$  and the lines  $y$  equals  $x$ . OK.

And now, if we move  $z$  away from that, then what we're getting are hyperbolas, and these hyperbolas will approach this asymptotic line  $y$  equals minus  $x$  or this line-- sorry, this line  $y$  equals  $x$  or this line equals  $y$  equals minus  $x$ . They'll approach this as they go down, but they'll never quite reach it. So the level curves here are just hyperbolas.

So now let's see, what is this telling us about the three-dimensional graph of this function? OK, so, first of all, we have these level curves when  $y$  equals  $x$  and when  $y$  equals minus  $x$ , and so

those level curves we can kind of draw. OK, so I want you to think that that sits in the  $xy$ -plane. It's kind of hard to draw in three dimensions. And so this is where our function is going to be zero.

Now, if we take  $x$  to be positive-- sorry. If we take  $x$  to be greater than  $y$  and both positive, then this is a positive number and this is a positive number. So if we look in the region where  $x$  and  $y$  are both positive, that's in here, and where  $x$  is greater than  $y$ , then our function comes up. So it looks like this, and then it dips down and goes down, comes back up, and goes back down. And now at the middle here, it has to dip down to zero, so we have something like this. So what we end up getting in the end, this is a saddle, so it's a bit hard to draw. It's a bit hard to see on this so let me draw a sketch of it off of the axes for you. So we have a rise, and then a drop, and then a rise in the back, and then a drop, and then down the middle it dips down in this direction and it rises up in this direction, so it's a saddle like you could put this on a horse and ride it.

And so we can see that the three-dimensional contours of the saddle, when we look at their projection down onto the contour plot, become these hyperbolas. So a saddle is sort of like a hyperbola stretched out into three dimensions. And I think I'll leave it at that.