

CHRISTINE

Welcome back to recitation. In this video, I'd like us to work on the following problem.

BREINER:

So the problem is as follows. For which of the following vector fields is the domain where each vector field is defined and continuously differentiable a simply connected region. So there's a lot there. I'm going to break that down first, and then show you the vector fields.

So we're starting with some different vector fields. And we want to determine first the domain for each vector field where it is both defined and continuously differentiable. And then once you've determined that domain, the next object is to determine whether or not that region is simply connected. So there are two parts for each of these problems.

So again, the first thing is you want to determine all of the values for which the vector field is both defined and continuously differentiable. You want to look at that region that contains all those values, and you want to determine if that region is simply connected. So there are four different vector fields, and I'll just point them out here. They're all in the plane.

So the first one is $\sqrt{x} \mathbf{i} + \sqrt{y} \mathbf{j}$. The second one is $\mathbf{i} + \mathbf{j}$, divided by the square root of $1 - x^2 - y^2$. The third one looks fairly similar to the second one, but it's $\mathbf{i} + \mathbf{j}$, divided by the square root of the quantity $x^2 + y^2 - 1$. And the fourth one is $\mathbf{i} + \mathbf{j}$ times the quantity natural log of r^2 -- natural log of $x^2 + y^2$.

So there are four different vector fields here. You have to do two things for each one. So find the domain where it's defined and continuously differentiable, and then determine if that domain is simply connected. So why don't you pause the video, work on these, and then when you're ready to see what I did, you can bring the video back up.

OK, welcome back. So we're going to do this problem one vector field at a time. We're going to take each vector field, I'm going to determine where it's defined and differentiable, and then I'm going to determine if that region is simply connected. So we're going to do each one separately.

So I'm going to start off with a, which was $\sqrt{x} \mathbf{i} + \sqrt{y} \mathbf{j}$. And I want to point out first, for the function f of x is equal to square root of x , it is defined for all x greater than or equal to 0, and it is differentiable for x greater than 0. I left out the T. There we go. So it is both defined

and differentiable when x is greater than 0. And now if I replace this with a y , the same thing is true for y greater than or equal to 0, and y greater than 0. So we know the region where this vector field is both defined and differentiable is when x is greater than 0 and y is greater than 0.

So we need the region-- let me draw it this way-- the region that is the first quadrant in the xy -plane. And it's not including the x -axis or the y -axis. So it's this full shaded region. OK, that is the region where it's defined and differentiable.

If I was going to write that precisely, I would say something like, all (x, y) with x greater than 0 and y greater than 0. Right? You need both. So it's exactly the (x, y) pairs with x positive and y positive, and that's the region. And now the question is, is that region simply connected?

Well, the way we think about simply connectedness, from what you've seen in class, is you want to show that if you take a closed curve that's contained in the region, that everything on the interior of that closed curve is also in the region. And you notice that is in fact true for this first quadrant. Any closed curve I draw that's in the region, all the points on the interior of the curve are also in the region. So this domain where it's defined and differentiable is simply connected.

So the first one, it is simply connected. All right. So that's part a.

Part b-- let me rewrite that one so we don't have to zoom over to the other side-- this was $i + j$ divided by the function square root of $1 - x^2 - y^2$. Well, we already know that the square root function is defined as long as the inside function is greater than or equal to 0. Because it's in the denominator, we actually need this function $1 - x^2 - y^2$ to be greater than 0.

And that's also where it's going to be differentiable. The differentiable and the defined regions are exactly the same, and they're both where $1 - x^2 - y^2$ is greater than 0. Right? The function is only defined as long as this quantity is greater than 0, and that's exactly where it's differentiable as well.

And so what does this correspond to? Well, if you think about it, this is actually $1 > x^2 + y^2$. And what are the points that look like this? Well, the x - and y -points that satisfy this inequality are the x - and y -values that are on the interior of the unit circle.

So if I draw a picture of that. Let me try and dot the unit circle. It's not containing the boundary, but it's all the points that are on the interior of the unit circle. Every point here, when I take the ordered pair (x, y) and it's on the interior of the unit circle, it satisfies this inequality. Those are the only points that do that.

So this is the region that has this vector field both differentiable and defined. Right? Now, is this region simply connected? It is, again for the same reason. Because if you take any closed curve here, and you look at the interior of that closed curve, every point on the interior of that closed curve is also in the region. Right? So it is also simply connected.

OK. So we had two so far that were simply connected. They were different-looking regions, but ultimately they both had any closed curve, the interior of it was all contained in the region that we were interested in.

So now the third one we have is somewhat similar-looking to part b, except that what's in the square root is a little different. So now we can use exactly the same logic as what we did in part b. And what we see is by the exact same logic that this vector field will be defined and differentiable as long as $x^2 + y^2 - 1$ is positive. Because that's where the square root function is differentiable, and that's also where 1 divided by the square root of this thing is defined.

So they correspond to exactly the same regions. And that is $x^2 + y^2 > 1$. So if you think about that, what we did previously was we had $x^2 + y^2 < 1$. So obviously if you want $x^2 + y^2 > 1$, we're taking all the (x, y) pairs that are outside the unit circle.

So again, we take the unit circle. We don't include the unit circle, because that's where $x^2 + y^2 = 1$. And then we want all of the values outside of that region. So this extends off to infinity. All of the values outside of that region.

Now, is this region simply connected? It is not. And the point is that while you do have some curves, that if I take a closed curve, all of the points on the interior of that closed curve are in the region, there are some curves for which that's not true.

For example, if I take the circle of radius-- what does that look like-- $2, 1$ and $1/2$, something like that? If I look at all of the points on the interior of this curve-- I'm going to try and shade it without getting rid of everything, so you can see still what's behind-- if I look at all those points,

notice in particular, there are a bunch of points-- for instance, this one here, this one here, and this one here-- all the ones inside the unit circle, are on the interior of this curve, but they're not in the region. Right?

I'll shade it extra dark. All the points that are in here, that are inside the unit circle, are actually still on the interior of this curve, but they're not in the region. And while there are some curves for which everything on the inside is in the region, but there are some curves for which it's not true, and that is what we know about not-simply connectedness. So we know this one is not simply connected. OK.

So now we have one left. And the last one was i plus j , times natural log of x squared plus y squared. So in this vector field, what I'm really interested in is the behavior of this function natural log of x squared plus y squared. And the point I want to make is that natural log is defined as long as the input value is positive, and it is differentiable everywhere it's defined. And so this function will be both defined and differentiable as long as x squared plus y squared is greater than 0.

So it's defined for all of these values, and natural log is differentiable everywhere it's defined. So it's also differentiable for all of these values. And so we see that this vector field is defined and differentiable everywhere except at one point.

And so let me draw-- this is a zooming in of that it's missing that point-- I want to make it extra large. But it's really only missing one point. And what point is that? That point is the origin. So everywhere except the origin. Maybe I should make it smaller, because maybe it looks like it's missing a whole circle.

It's just missing the point. It's just missing the origin. But every other point on the xy -plane is a place where this vector field is differentiable and defined. Right? So it's only missing that one point, but that still gives us the fact that this region is not simply connected.

And again, it's exactly the same type of logic as the previous problem, that I could draw curves, where every point on the interior is contained in the region. But there are curves that also fail. Right? If I draw a curve that contains the origin, every point on the interior of this region except the origin, is contained in the domain of interest, right? But because the interior of the curve contains the origin, I know that this region is, in fact, not simply connected.

So if I look at all of \mathbb{R}^2 -- so if I look at all the (x, y) -values except x equals 0 and y equals 0--

except the origin-- I get a region that's not simply connected. And this one is maybe a little tricky, so I'm going to say it one more time. OK?

So while there are some curves that we can see some portions of this region behave like simply connected regions, when you're around the origin, any curve you take around the origin is going to contain the origin on its interior. Right? But the origin is not in our domain of interest. And therefore, there are curves that we can take that their interior contains a point that's not in the region. And that's what it means to be not simply connected. OK?

So let me go back to the beginning and just remind you again real quickly what we did here. We had these four vector fields. We wanted to do two things with all of them. We wanted to first find the regions where they were defined and differentiable, and then determine if those regions were simply connected.

And so we had two examples where the regions were simply connected. And then I gave you two examples where the regions were not simply connected. And so hopefully this was informative for how we can understand that vector fields are not necessarily always defined everywhere, but also to understand what this simply connected region term actually means. And I guess that's where I'll stop.