

PROFESSOR: Hi, welcome back to recitation. I have a nice exercise here for you that tests your knowledge of triple integration. So in particular, I've got for you a cylinder. And my cylinder has height h and it has radius b . And this is the kind of cylinder I like. It's a constant density cylinder. So its density is just 1 everywhere. So what I'd like you to do is, for the cylinder, I'd like you to compute its moment of inertia around its central axis. So why don't you pause the video, have a go at that, come back, and you can check your work against mine.

Hopefully you had some luck working on this problem. Let's talk about it. So the first thing to notice is that there aren't any coordinates in this problem. I've given you a cylinder but it's up to you to choose coordinates, a way to arrange your cylinder in space or a way to arrange your coordinates with respect to your cylinder.

So a convenient thing to do in this case is going to be-- you know, we're working with respect to the central axis of the cylinder. So let's make that one of our axes. So in particular, why don't we make it our z -axis. That seems like a natural sort of thing to do. So let me try and draw a little picture here. So we've got our cylinder. And there it is with our three coordinate axes. I guess it's got radius b and it's got height h .

Now we've arranged it, now we have coordinates, so now we want to see what it is we're trying to do with it. So we're trying to compute a moment of inertia. So we have to remember what a moment of inertia means. So let me think. So a moment of inertia, when you have a solid-- so your moment of inertia I with respect to an axis is what you get when you take the triple integral--

so let's say your solid is D . Your solid D . So you take D . So you take a triple integral over D and you're integrating r squared with respect to the element of mass. OK. So r squared here, this is the distance from the axis around which you're computing the moment of inertia. And in our case, so in any case, this little moment of mass is-- sorry, little element of mass-- is density times a little element of volume. So we can also write this as the triple integral over our region of r squared times δ times dV .

OK, so this is what this is in general. So now let's think about it in our case. Well, in our case, we've cleverly chosen our central axis to be the z -axis. So this r is just the distance from the z -axis. This I was kind enough to give you a constant-density cylinder, so this δ is just 1. That's going to be easy. And then we have this triple integral over the whole cylinder. So we

want a triple integral over the whole cylinder of some integrand that involves r .

So a natural thing to do in this situation-- the natural sort of thing to do when you have a cylinder or anything that's rotationally symmetric, and you have an integrand that behaves nicely with respect to rotation, that can be written easily in terms of r , or r and θ -- is to do cylindrical coordinates. Is to think of cylindrical coordinates. So in our case that means we just need to figure out-- at this point-- we need to figure out how do we integrate over the cylinder in cylindrical coordinates?

So let's do it. So in our case-- so it doesn't matter too much what order we do things in. So we need dV . We need to write that in terms of the cylindrical coordinates. So that's dz , dr , and $d\theta$. And so we know that dV is $r dz dr d\theta$. You might want some other order there, but that's a good, nice order. It usually is the simplest order to consider. So this moment of inertia, in our case, is going to be this triple integral.

OK, so we said $r^2 \delta$, r^2 times density, density is 1. So that's just r^2 . And r , the distance to the axis is r , the distance to the z -axis. So that's just r^2 times $r dz dr d\theta$. So this is the integral we're trying to compute, but we need bounds, right? It's a triple integral, it's a definite integral, we need to figure out what the bounds are to evaluate it as an iterated integral.

So let's go look at this little picture we drew. So I guess I didn't discuss this, but I made a choice just to put the bottom of the cylinder in the xy -plane and the top at height h . It's not going to matter. If you had made some other choice, it would work out fine. So that means the z is going from 0 to h , regardless of r and θ . So z is going from 0 to h . That's nice, let's put that over here. z is the inside one. It's going from 0 to h . Then r is next. Well, this is also-- you know, cylinders are great for cylindrical coordinates. Shocker, right, given the name. I know.

So r is going from 0 to what's the radius? Our radius was b . So r goes from 0 to b , and that's true regardless of θ . So back over here, so we have r going from 0 to b and θ is just going from 0 to 2π ; we're doing a full rotation all the way around the cylinder. So this is what our moment of inertia is, and now we just have to compute it.

So we've got our inner integral here is with respect to z . So the inner integral is the integral from 0 to h of $r^3 dz$. And r^3 doesn't have any z 's in it. Fabulous. So that's just going to be $r^3 z$, where z goes from 0 to h . So that's $h r^3$ minus 0. So $h r^3$. All right. So that's the inner.

Now let's look at the middle integral. So this is going to be the integral as-- that's our r integral. So that's going from 0 to b . And what are we integrating? We're integrating the inner integral. So the inner integral was $h r^3$. So we're integrating $h r^3 dr$ from 0 to b . All right, this is not quite as easy. But h as a constant, we're integrating r^3 . I've done worse. You've done worse. So that's going to be $h r^4/4$ between 0 and b . So OK. So that's $h b^4/4$ minus $h \cdot 0^4/4$. So the second term's 0. So this is just equal to $h b^4/4$.

And finally, we have our outermost integral. So what was that integral? Well, that was the integral from 0 to 2π $d\theta$ of the second integral. So this is of the middle integral. So it's the integral from 0 to 2π $d\theta$ of the middle integral which was $h b^4/4$. And this is just a constant again. Great. So this is $h b^4/4$ times 2π . So what does that work out to? That's $h b^4 \pi/2$.

All right. So there you go. Now if you wanted to, you could also rewrite this a little bit, because you could note that this is $\pi h b^2$, that's your volume of your cylinder. And in fact, it's not just your volume, it's your mass of your cylinder, because it had constant density 1. So you also could've written this as mass times $b^2/2$. Sorry. $b^2/2$. I don't know where a came from. Mass times b^2 . So you have some other options for how you could write this answer by involving the volume and mass and so on.

So let's just recap very quickly why we did what we did. We had a cylinder. And so really, given a cylinder, it was a natural choice to look at cylindrical coordinates. And once we had cylindrical coordinates, everything was easy. So we just took our general form of the moment of inertia, took the region in question, in cylindrical coordinates it was very, very easy to describe this entire region. And then our integrals were pretty easy to compute after we made that choice. After we made that choice they were nice and easy to compute. So I'll stop there.