

18.02 Problem Set 12

At MIT problem sets are referred to as 'psets'. You will see this term used occasionally within the problems sets.

The 18.02 psets are split into two parts 'part I' and 'part II'. The part I are all taken from the supplementary problems. You will find a link to the supplementary problems and solutions on this website. The intention is that these help the student develop some fluency with concepts and techniques. Students have access to the solutions while they do the problems, so they can check their work or get a little help as they do the problems. After you finish the problems go back and redo the ones for which you needed help from the solutions.

The part II problems are more involved. At MIT the students do not have access to the solutions while they work on the problems. They are encouraged to work together, but they have to write their solutions independently.

Part I (20 points)

At MIT the underlined problems must be done and turned in for grading.

The 'Others' are *some* suggested choices for more practice.

A listing like '§1B : 2, 5b, 10' means do the indicated problems from supplementary problems section 1B.

1 Line integrals and Gradient fields in 3D. Curl, exactness, and potential. 3-space.

§6D: 1a, 2, 4, 5; Others: 1bcd

§6E: 3ab(ii)(both methods), 5; Others: 1, 2, 4

2 Stokes' Theorem.

§6F: 1b, 2, 5; Others: 1a, 3

3 Stokes' Theorem – theory, interpretation, applications.

§6G: Others: 1, 2

§6H: Others: 1, 2

Part II (20 points)

Problem 1 (5: 1,2,1,1)

Let $\mathbf{F}(x, y, z) = \left(\frac{-z}{x^2 + z^2}\right) \mathbf{i} + y \mathbf{j} + \left(\frac{x}{x^2 + z^2}\right) \mathbf{k}$ defined for all points (x, y, z) in 3-space not on the y-axis (that is, all points for which $x^2 + z^2 > 0$).

a) By direct computation, show that $\nabla \times \mathbf{F} = \mathbf{0}$ for all points not on the y-axis.

b) By direct computation, show that $\oint_{C_1} \mathbf{F} \cdot d\mathbf{r} = 0$ where C_1 is the closed curve defined by $x^2 + y^2 = 1, z = 1$.

c) Can you use Stokes' Theorem and the fact (from part (a)) that $\nabla \times \mathbf{F} = \mathbf{0}$ to conclude that $\oint_{C_2} \mathbf{F} \cdot d\mathbf{r} = 0$ when C_2 is the closed curve defined by $x^2 + z^2 = 1, y = 0$? Why/ why not?

d) Compute out $\oint_{C_2} \mathbf{F} \cdot d\mathbf{r}$ and see what happens.

Problem 2 (5: 2,2,1)

Suppose the field \mathbf{F} in the previous problem is replaced by the field

$$\mathbf{G}(x, y, z) = \left(\frac{x}{x^2 + y^2 + z^2}\right) \mathbf{i} + \left(\frac{y}{x^2 + y^2 + z^2}\right) \mathbf{j} + \left(\frac{z}{x^2 + y^2 + z^2}\right) \mathbf{k}$$

defined for all points $(x, y, z) \neq (0, 0, 0)$.

a) Show that $\nabla \times \mathbf{G} = \mathbf{0}$ for all points $(x, y, z) \neq (0, 0, 0)$

b) Can you use Stokes' Theorem in this case and the fact that $\nabla \times \mathbf{G} = \mathbf{0}$ for all points $(x, y, z) \neq (0, 0, 0)$ to conclude that $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ for all simple closed curves C which do not pass through the origin?

c) Explain the difference between these two cases in term of the connectedness type of the domains of definition of the two fields.

In problems 3 and 4, we have a non-steady flow $\mathbf{F}(x, y, z, t) = \rho(x, y, z, t) \mathbf{v}(x, y, z, t)$ given by

$$\mathbf{F}(x, y, z, t) = (z \sin t) \mathbf{i} + (-z \cos t) \mathbf{j} + (-x \sin t + y \cos t) \mathbf{k}.$$

We'll take the case where the density $\rho(x, y, z, t) = 1$, constant, so that $\mathbf{F}(x, y, z, t) = \mathbf{v}(x, y, z, t)$ as vectors (although in different physical units).

In general for non-steady flows, the divergence $\nabla \cdot \mathbf{F}$ of the field $\mathbf{F}(x, y, z, t)$ is defined using differentiation $\nabla = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle$ with respect to the *space* variables only. For this reason, at any fixed time t the Divergence Theorem holds, and the physical interpretations of the divergence $\nabla \cdot \mathbf{F}$ of the field and of the flux of \mathbf{F} through a surface are the same as the ones given in the Notes §V10 for the case of steady flows. By

the same token, Stokes' Theorem holds at each fixed time t , and hence the physical interpretation of the curl $\nabla \times \mathbf{F}$ is the same as that given in the Notes §V13 (pp.3-4) for the case of steady flows.

Problem 3 (2: 1,1)

- a) Show that this flow \mathbf{F} satisfies the *equation of continuity* $\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{F} = 0$.
- b) Referring to the facts about non-steady flows given above: what is the net outward flux of \mathbf{F} through any simple closed surface, and why?

Problem 4 (8: 2,1,2,2,1)

- a) Compute the curl $\nabla \times \mathbf{F}$ of \mathbf{F} .
- b) Referring to the facts about non-steady flows given above: at any fixed time t and position (x, y, z) , what is the axial direction perpendicular to the plane in which the flow produces the maximum angular velocity, and what is the maximum angular velocity? Does it vary at the different points (x, y, z) in space?
- c) Show that at any time t : (i) the velocity vectors $\mathbf{v}(x, y, z, t)$ of the flow lie in the plane through the origin with normal $\mathbf{n}_t = \langle \cos t, \sin t, 0 \rangle$ (which we'll call \mathcal{P}_t); and (ii) the velocity vectors $\mathbf{v}(x, y, z, t)$ are perpendicular to the radial vectors $\mathbf{r} = \langle x, y, z \rangle$.
- Combining the result of parts(b) with (i): what is the plane of greatest spin for this flow?
- d) Use the results of part(c) to give a sketch of some representative velocity vectors in the plane \mathcal{P}_t at a fixed time t . (Take t to be about $\frac{\pi}{4}$, and first sketch \mathcal{P}_t in 3-space; then for the purposes of sketching the velocity field, you can take \mathcal{P}_t to be the plane of the paper.)
- e) Putting together the results above, describe in general terms the pattern of the motion of the flow in 3-space over time.

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