

Welcome back to recitation. In this video I'd like us to consider the following problem. So the question is for what simple closed curve C , which is oriented positively around the region it encloses, does the integral over C of minus the quantity $x^2y + 3x - 2y$ $dx + 4y^2x - 2x$ dy achieve its minimum value?

So again, the thing we want to find here, the point we want to make, is that we have this integral and we're allowed to vary C . So we're allowed to change the curve, the simple closed curve. And we want to know for what curve C does this integral achieve its minimum value? So why don't you work on that, think about it, pause the video, and when you're feeling like you're ready to see how I do it, bring the video back up.

Welcome back. So again, what we would like to do for this problem is we want to take this quantity, which is varying in C , and we want to figure out a way to minimize it. To find its minimum value. And actually what's interesting to think about, before we proceed any further, is you might think you want to take the smallest possible simple closed curve you can. In other words, you might want to shrink it down to nothing. But then this integral would be 0 and the question is, can we do better? Because we could have a minimum value, of course, that's negative. And so we could do better by having a larger curve put in the right place.

So I just want to point that out, if you were thinking "I'll just make C not actually a curve, I just shrink it down to nothing." That won't be the best we can do. We can actually find a curve that has an integral that is not 0, but is in fact negative. And then we want to make it the most negative we can. So that's the idea.

So what we're going to do here is we're going to use Green's theorem to help us. And so what we want to remember is that if we have the integral over C of $M dx + N dy$ we want to write that as the integral over R of $N_x - M_y$ $dx dy$. So what I want to point out, I'm just going to write down what these are. I'm not going to take the derivatives for you. I'm just going to show you what they are.

So in this case, with the M and N that we have, we get exactly this value for the integral. We get $N_x - M_y$ is equal to $x^2 + 4y^2 - 4$ $dx dy$. You can obviously compute that yourself. Just take the derivative of N with respect to x , subtract the derivative of M with respect to y , you get a little cancellation and you end up with this.

And so now, instead of thinking about trying to minimize this quantity over here in terms of a curve, now we can think about trying to minimize the quantity here in terms of a region. And the goal here is to make the sum of all of this over the whole region-- we want to make it as negative as we can possibly make it. So essentially what we want to do is we want-- on the boundary of this region, we would like the value here to be 0, and on the inside of the region we'd like it to be negative.

So let me point that out again and just make sure we understand this. To make this quantity as small as possible, what we would like-- let me actually just draw a little region. So say this is the region. To make this integral as small as possible, what we want is that $x^2 + 4y^2 - 4$ is negative inside the region. So if this whole quantity is less than 0 inside the region, and we want it to, on the boundary of the region, equal 0. And why do we want it to equal 0 on the boundary? Well, that's because then we've gotten all the negative we could get and we haven't added in any positive and brought the value up. So that's really why we want the boundary of the region to be exactly where this quantity equals 0.

And so let's think about, geometrically, what describes R -- oops, that should have an s . What describes R , where $x^2 + 4y^2 - 4$ is less than 0 inside R , and that's the same thing as-- in what we're interested in-- $x^2 + 4y^2 - 4 = 0$ on the boundary of R .

And if you look at this, this is really the expression that will probably help you. This expression, if you rewrite it, you rewrite it as $x^2 + 4y^2 = 4$. And you see that this is actually the equation that describes an ellipse. Maybe you see it more often if you divide everything by 4, and so on the right-hand side you have a 1, and your coefficients are fractional, potentially, there. But this is exactly the equation for an ellipse. And so the boundary of R is an ellipse described by this equation, but the boundary of R is actually just C .

So C we now know is exactly the curve that is carved out by this equation on the plane, on the xy -plane. That's an ellipse. So just to remind you what we were trying to do. If you come back here, we were trying to figure out a way to minimize the certain integral over a path. And what we did was instead of trying to look at a bunch of paths and figure out what would minimize that, we tried to see if Green's theorem would help us. So Green's theorem allowed us to take something that was an integral over a path and change it to an integral over a region.

And then when we look at what we ended up with, we realize that we could make this the most minimum if we let this be on the region where it was negative everywhere. So we were looking for a region where this quantity was everywhere negative on the inside and 0 on the boundary. And that's exactly what we did. And then we see that we get to a point where the boundary has this equation, $x^2 + 4y^2 = 4$. We see then the boundary's an ellipse, and C is indeed the boundary of the region. So we see that C is the ellipse described by this equation. So that's where I'll stop.