

JOEL LEWIS: Hi. Welcome back to recitation. In lecture, you've learned about gradients and about directional derivatives. So I have a problem here to test your understanding of those objects and give you some practice computing them.

So I have three functions here. And for each function, I've given you a point and a direction. So what I'd like you to do is compute the gradients of the functions, then evaluate their gradients at the point given, and then compute the directional derivative of the function at the point in the direction of the vector given.

So here, for example, in part a, I've given you the function f of x, y equal to $x^2y + xy^2$, at the point x equal minus 1, y equal 2, and in the direction of the vector $[3, 4]$. And then we've got two more examples here. g of x, y, z is equal to the square root of $x^2 + y^2 + z^2$, P is the point 2, 6, minus 3, and v is the direction $[1, 1, 1]$. And our third example, h is a function of four variables, $w, x, y,$ and z . It's given by $w^2x + w^2y + w^2z + x^2y + x^2z + y^2z$, at the point 2, 0, minus 1, minus 1, in the direction 1, minus 1, 1, minus 1.

So why don't you pause the video, take some time to work out these three problems, come back, and we can work them out together.

I hope you had some luck working out these problems. Let's get started on the first one.

So on the first one, our function f of x, y is equal to $x^2y + xy^2$. So to compute the gradient, I just have to compute the first partials of our function, and then put them into a single vector. So that's what the gradient is.

So we have that the gradient of f at (x, y) is equal to-- it's the vector, and its first component is the first partial of f with respect to x , so that's going to be $2x^2y + y^2$. And its second component is the partial of f with respect to y , so that's going to be $x^2 + 2x^2y$. So this is the gradient of our function f .

Now, the question asked you to compute this gradient at the particular point minus 1, 2. So we have gradient of f at the point minus 1, 2 is equal to-- well, we just put in x equal minus 1, y equal 2 into this formula. So at x equal minus 1, y equals 2, this is 2 times minus 1 times 2 plus 2 squared. So that's minus 4 plus 4, so that's just 0. And the second term is minus 1 squared plus 2 times minus 1 times 2. So that's 1 plus 4-- nope-- 1 plus negative 4, so that's negative

3. All right. So this is the gradient of our function f . This is the gradient at the point minus 1, 2.

Now we are asked for the directional derivative. So we were asked for the directional derivative of f in a direction. So we're given-- I didn't give you-- so I gave you a vector v . But when we take directional derivatives, what we want is a unit vector. Right? So the vector v that I gave you-- was right over here-- it was this vector, $[3, 4]$. And so we need to find the unit vector u in the direction of v in order to apply our usual formula.

So u is just v divided by the length of v . So in this case, the vector $[3, 4]$, by the Pythagorean theorem, it has length 5. So this is equal to the vector $3/5$ comma $4/5$. And so our directional derivative in the direction of v -- which is this vector u -- it's just equal to the gradient at that point dotted with the direction vector.

So this is equal to gradient of f at that point dot u . Which in our case is equal to-- well, since we're interested at the point P -- it's equal to 0, minus 3, dot our vector u , which is $[3/5, 4/5]$. And that dot product is negative $12/5$. So in this direction, the function is decreasing at about this rate at the given point.

All right. That's part a. Let's go on to part b. So, you know, you've probably noticed that these questions are all fairly similar. The only real complication is that I've been increasing the number of variables. But of course, you know how to take a gradient for a function of three variables as well.

So in this case, our function g of x, y, z is given by the square root of x squared plus y squared plus z squared. So, if you like, this is the function that measures the distance of a point from the origin. And so we're asked first for the gradient, so the gradient of g . What do I do?

Well, again, I just take these partial derivatives. This is going to be a tiny bit messy, I'm afraid. Because I take a partial derivative of this expression with respect to x , I've got to apply the chain rule, right? Because this is a composition of functions here. So OK. So, what do I get?

I get the derivative of the inside times the derivative of the out. So the outside function is a square root, so that's to the one halfth power. So I get $1/2$ times the thing inside to the minus one halfth power. So it's over 2 times the square root of x squared plus y squared plus z squared. And then I need to multiply by the derivative of the inside, which is times $2x$. OK. So, and then the 2's cancel, and that's x over square root of x squared plus y squared plus z squared.

And similarly, this is a nice symmetric function, and, you know, if I changed x and y , it's the same. So the other partial derivatives look very similar.

So they're going to be y over the square root of x squared plus y squared plus z squared. And z over the square root of x squared plus y squared plus z squared. Sorry. That's a little bit of a long formula there. But there we have it. That's the derivative-- or, sorry, not the derivative. That's the gradient. The vector of partial derivatives.

So now you were asked also to compute this gradient at a particular point. So the point in question-- I have to look back over here. And the point in question was this point, 2, 6, minus 3. So at the point 2, 6, minus 3, we want to compute the gradient. So we just take these numbers back over to our formula over there, and we're going to put them in.

So we take the gradient of g at 2, 6, minus 3. Well, OK, so this square root of x squared plus y squared plus z squared appears in all terms. So let's compute that first.

So x squared is 4, y squared is 36, and z squared is 9. So I add those numbers together, I get 49. And then I take a square root of that and I get 7. So these denominators are all going to be 7, and then up top I just have x , y , and z . So this is going to be 2 over 7 comma 6 over 7 comma minus 3 over 7.

All right. So just what I get by plugging the values at our point into this formula. So this is the gradient at that point. And now once again, I want to compute a particular directional derivative of this function. So in order to do that, I just need the right unit vector, and I need to dot it.

So the direction that I asked you was the direction v with coordinates 1, 1, 1. So of course again, this isn't a unit vector. So we need to divide it by its length to find the unit vector that we're going to use.

So the length of this vector is the square root of 1 squared plus 1 squared plus 1 squared, so that's the square root of 3. So u is equal to-- well, I'm just going to write it as 1 over the square root of 3 times the vector $[1, 1, 1]$.

And so the directional derivative dg/ds in the direction u hat is-- again, it's what I get, I dot the gradient with this u -- it's gradient dot u . Which in our case, so that 1 over the square root of 3 lives out front. And now I dot 2/7, 6/7, minus 3/7 with $[1, 1, 1]$. And that's just going to give me 2/7 plus 6/7 minus 3/7. And OK, so we can finish off the arithmetic there, and I think that's 5

over $7\sqrt{3}$ if I didn't make any mistake.

Now, the last one, very, very similar. Now we have a function of four variables. There's really nothing new there at all. So let me rewrite. So for part c, we have h of w, x, y, z is equal to w^2x plus w^2y plus w^2z plus x^2y plus x^2z plus y^2z . So, for the gradient of h , I just take the four partial derivatives.

Now here, w is first. So I take the partial derivative with respect to w as the first coordinate. So that's going to be x plus y plus z . And then I take the partial derivative with respect to x as the second coordinate. So that's going to be w plus y plus z . And similarly, the last two are w plus x plus z , and w plus-- that's a plus-- x plus y . Whew. OK.

So that's my gradient. Those are my partial derivatives. Now, I asked you again for the gradient at a particular point.

So that's $\text{grad } h$ at the point $2, 0, -1, -1$. So we can just plug that in. Here, it's 0 plus -1 plus -1 is -2 . Here it's 2 plus -1 plus -1 , that's 0 . Here, it's 2 plus 0 plus -1 so that's 1 . And lastly, it's 2 plus 0 plus -1 is 1 . So that's the gradient of this function h at this point P .

And now I gave you again, I gave you a vector v . So this was the vector $1, -1, 1, -1$. And I asked you to find the directional derivative at this point in that direction. Again, we need a unit vector. This isn't a unit vector.

So u is equal to v over the length of v . Well, what is the length of v in this case? It's over the square root of-- well, we square the coordinates and add them. So that's 1 plus 1 plus 1 plus 1 . So that's 4 ; square rooted is 2 . So that's v over 2 .

And finally, I get that the directional derivative dh/ds in this direction u , is what I get when I dot this gradient together with the direction in which I'm heading. So this is $-2, 0, 1, 1$. The gradient dotted with the unit vector of the direction. Which we just computed.

And so finally, you could expand this out yourself. I won't write down the final answer, but it's just a dot product, right? All right.

So hopefully this has gotten you totally comfortable with computing gradients and with computing directional derivatives from gradients. So I'll end there.