

## Speed and arc length

1. A rocket follows a trajectory

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} = 10t\mathbf{i} + (-5t^2 + 10t)\mathbf{j}.$$

Find its speed and the arc length from  $t = 0$  to  $t = 1$ .

**Answer:**

$$\text{velocity} = \mathbf{v}(t) = \frac{d\mathbf{r}}{dt} = 10\mathbf{i} + (-10t + 10)\mathbf{j} \Rightarrow \frac{ds}{dt} = \sqrt{10^2 + (-10t + 10)^2} = 10\sqrt{1 + (1 - t)^2}.$$

$$\text{Arc length } L = \int_0^1 \frac{ds}{dt} dt = 10 \int_0^1 \sqrt{1 + (-t + 1)^2} dt$$

Make the change of variables  $u = -t + 1$

$$\Rightarrow du = -dt, t = 0 \rightarrow u = 1, t = 1 \rightarrow u = 0. \Rightarrow L = 10 \int_0^1 \sqrt{1 + u^2} du.$$

We can compute this integral with the trig. substitution  $u = \tan \theta$  or by use of tables

$$\Rightarrow L = 10 \int_0^{\pi/4} \sec^3 \theta d\theta = 5 [\sec \theta \tan \theta + \ln(\sec \theta + \tan \theta)]_0^{\pi/4} = 5(\sqrt{2} + \ln(\sqrt{2} + 1)).$$

2. For the cycloid  $x = a\theta - a \sin \theta$ ,  $y = a - a \cos \theta$  find the velocity, speed, unit tangent vector and arc length of one arch.

We will use the trigonometric formulas

$$\sin^2(\theta/2) = \frac{1 - \cos \theta}{2} \quad \text{and} \quad \sin \theta = 2 \sin(\theta/2) \cos(\theta/2).$$

Computing,

$$\frac{d\mathbf{r}}{d\theta} = a(1 - \cos \theta, \sin \theta) = 2a(\sin^2(\theta/2), \sin(\theta/2) \cos(\theta/2)),$$

which implies

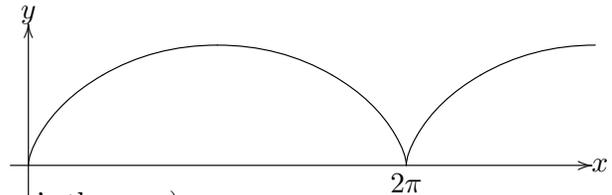
$$\left| \frac{d\mathbf{r}}{d\theta} \right| = \frac{ds}{d\theta} = 2a \sqrt{\sin^2(\theta/2)} = 2a |\sin(\theta/2)|.$$

$$\text{So, } \mathbf{T} = \frac{2a(\sin^2(\theta/2), \sin(\theta/2) \cos(\theta/2))}{2a |\sin(\theta/2)|} = \pm (\sin(\theta/2), \cos(\theta/2)) \quad (\text{a unit vector!})$$

Note, at the cusp ( $\theta = 2\pi$ )  $ds/d\theta = 0$ , i.e., you must stop to make a sudden 180 degree turn.

For one arch,  $0 < \theta < 2\pi$ ,  $\frac{ds}{d\theta} = 2a \sin(\theta/2)$

$$\begin{aligned} \Rightarrow \text{arc length} &= \int_0^{2\pi} \frac{ds}{d\theta} d\theta \\ &= \int_0^{2\pi} 2a \sin(\theta/2) d\theta \\ &= -4a \cos(\theta/2) \Big|_0^{2\pi} \\ &= 8a \quad (\text{this is sometimes called Wren's theorem}). \end{aligned}$$



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18.02SC Multivariable Calculus  
Fall 2010

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