

18.02SC Multivariable Calculus, Fall 2010  
Transcript – Recitation 24, Max/Min

JOEL LEWIS: Hi. Welcome back to recitation. One of the things you've been learning about in lecture is how to solve some max-min problems. How to find the maximum or minimum of a given function given some constraints and figure out where it reaches its largest or smallest value. So here I have a particular example of such a problem.

So we're going to build a cardboard box. And our cardboard box has to meet the following criteria. So its volume has to be 3 units. The front and back of the cardboard box are going to be made just of single-thickness cardboard. But the two sides, the left and right, are going to be made double-thick. And the bottom is going to be made triple-thick. We're not going to have a top, it's just going to be an open box. So it's going to have five sides: two of single thickness, two of double thickness, and with the bottom of triple thickness.

So, the question is-- there are, you know, a lot of different shapes of box with total volume 3 units. So the question is what dimensions should this box have so that we use the minimum amount of cardboard possible? All right? So we want the dimensions of the box that use the least total amount cardboard.

So why don't you pause the video, work on that for awhile, come back, and we can work on it together

I hope you had some fun puzzling this one out. Let's have a go at it. So to start, I think I'd just like to draw a little picture of a box.

So here's my cardboard box. Its top is going to be open, say. So let's give it-- we want to figure out what its dimensions should be in order to minimize some expression, so let's give them names. Right? So I think a natural thing to call them is we can call one of them  $x$  and one of them  $y$  and the height  $z$ . There.

So if we give the box these dimensions  $x$ ,  $y$ , and  $z$ , we need to ask, well first of all, what is its volume? And second of all, how much cardboard is used? And as long as we can keep the volume equal to 3, what's the least amount of cardboard we can use?

So the volume of this box is just  $xyz$  and that has to be 3. So we know  $xyz$  is equal to the volume, which is equal to 3. So we have a constraint on  $x$ ,  $y$ , and  $z$  here. And the thing that we want to optimize-- that we want to find a minimum for-- is the total amount of cardboard used. So let's talk about how much cardboard is used.

So the bottom of the box is triple-thick. So its area is  $x$  times  $y$ . So the total amount of cardboard used is  $3xy$ .

Now, the front and back of the box are single thickness. And they have area  $xy$ , but there-- sorry, not  $xy$ ,  $xz$ , right? This height is  $z$ , so it's  $x$  times  $z$ -- and there are two of them. So we have  $xz$  total cardboard in front and  $xz$  total cardboard in back. Oh, I guess this has another segment there in back because it's an open box. So  $xz$  and  $xz$ . So that's  $2xz$  coming from the front and back.

And from the two sides, well each side has area  $yz$ . There are two of them and they're double-thick. So that contributes  $4yz$  to the total amount of cardboard used.

So this is the total amount of cardboard used. And we know  $xyz$  is equal to 3. So we want to minimize this, but we want to minimize this taking this into account, taking this restriction on the volume into account.

So what we can do is you can realize that we can just eliminate one of these variables. Right? We can say  $z$  is equal to  $3$  over  $xy$ . That always has to be true. And if we make that substitution in here, then we'll be able to minimize this expression without regard to any constraint anymore.

So from  $xyz$  equals 3, we have  $z$  equals  $3$  over  $xy$ . So we want to minimize what we get when we plug  $z$  equals  $3$  over  $xy$  into here. So that's  $3xy$ , plus-- well,  $2x$  times  $3$  over  $xy$  is--  $6$  over  $y$ , plus-- and  $4y$  times  $3$  over  $xy$  is--  $12$  over  $x$ .

And one other thing I guess we haven't mentioned explicitly is that this is a physical box. It has actual dimensions, so its dimensions have to be, you know-- they're lengths, they have to be positive numbers. So we want for  $x$  to be positive and we want for  $y$  to be positive numbers. OK.

So now we've got this function. And there aren't--  $x$  and  $y$  can be anything, any value of  $x$  and  $y$  we choose. This describes the amount of cardboard used in a box with those dimensions on its base that has volume 3. All right? So we want to minimize that. So now we've finally got to the point of our calculus. Let's call this function-- let's give it a name like, I don't know--  $f$  of  $x$ ,  $y$ .

So we want to find the global minimum of  $f$ . We want to find the absolute least amount of cardboard that we can use. So there are several possibilities for where a global minimum can occur. It can occur on some critical point of the function, or it can occur on the boundary of the region where the function is defined. And in this case, that includes the possibility that it can occur as  $x$  or  $y$  or both go off to infinity. All right?

So we have to look at those possibilities. In particular, we have to look at the critical points. That's one of the possibilities. So the critical points of  $f$ . So we need to find out what those points are.

So in order to find the critical points, well what do we do? We do the usual thing. We look at its partial derivatives. So we need the first and second-- or sorry-- the first partial derivatives with respect to  $x$  and  $y$ , we need both of them to be equal to 0. So we need  $f_x$  equals 0, and  $f_y$  equals 0. OK, so let's write down what that means over here.

So what is  $f_x$ ? Well, we just take the partials. So it's  $3y$ -- plus we take the partial of  $6y$  with respect to  $x$  and we get 0, and we take the partial of  $12$  over  $x$  with respect to  $x$ -- and we get minus  $12$  over  $x$  squared. So that's  $f_x$ , and we need that to be equal to 0. And  $f_y$  is equal to  $3x$  minus  $6$  over  $y$  squared, and we also need that to be equal to 0.

So if we solve this first equation, for example, we can solve it for  $y$  in terms of  $x$ . And that tells me that  $y$  is equal to-- I need-- let's see--  $12$  over  $x$  squared divided by 3, so that's--  $4$  over  $x$  squared. Now I can plug this  $4$  over  $x$  squared in down here, and so I get  $3x$  minus  $6$ . Well,  $6$  divided by  $4$  divided by  $x$  squared quantity squared is  $6$  times  $x$  squared over  $4$  squared. That has to be equal to 0.

And I can-- let's see- I can rewrite this. Maybe I'll divide through by 3 as that 3 isn't going to matter. So it's  $x$  minus-- so then I'll be left with  $2$  times  $x$  to the fourth over  $16$ , so that's--  $x$  to the fourth over  $8$  equals 0. And you can see in this equation that either  $x$  is equal to 0, or we can divide by  $x$  and then we get  $x$  cubed equals 8. So  $x$  equals 0 or  $x$  cubed equals 8. The only solution is  $x$  equals 2.

Now, it's easy to see that we can't have a box with  $x$  equal to 0, right? Our function is not actually defined at  $x$  equals 0 over here, you'll see, right. Our function had a  $12$  over  $x$  in it, so we can't have  $x$  equal to 0. So that's not going to be a critical point of this function.  $x$  equals 0 isn't going to lead to a critical point, so, OK, so our only critical points are going to happen when  $x$  is equal to 2.

And when  $x$  is equal to 2, we can go back up here and we see that  $y$  was equal to 4 over  $x$  squared at our critical points. So the only critical point is 2, 1. OK. So that's our critical point.

So now, we want to think about is this point going to be a maximum or a minimum or what? And where is the global minimum of this function going to occur? So remember that the global minimum of this function can occur either on the critical points or on the boundary or at a point where  $x$  or  $y$  is going to infinity-- in the limit, I guess I mean. So let's go back here and take a look at our function here.

So we can see from the expression for this function that if  $x$  and  $y$  are going to infinity, this is no good. This part is going to go to infinity.  $3xy$  is going to go to infinity, and these are going to be positive, so  $f$  is going to go to infinity. OK, so as the dimensions of our box get very, very large, the total amount of cardboard used becomes infinite.

Also, if  $x$  or  $y$  gets closer and closer to 0-- that's at the boundary-- well, if  $x$  gets closer and closer to 0, then  $12$  over  $x$  gets closer and closer to infinity. And if  $y$  gets closer and closer to 0, then  $6$  over  $y$  gets closer and closer to infinity. And again, all these terms are going to be positive because  $x$  and  $y$  have to be positive.

So in those limiting cases, the function  $f$  of  $x$ ,  $y$  gets infinitely large. So along the boundary and as  $x$  and  $y$  go to infinity, this function blows up. It gets very, very big. So that means the only place it could have a global-- so it has a global minimum-- the only place that global minimum can be is at a critical point. And we saw, if we walk back over here again, that the only critical point is the point 2, 1. All right? So that means this point has to be a global minimum for the function.

Now, you might, out of curiosity, you might ask how much cardboard are we actually talking about in that case? Since that's really the one part, I guess wasn't really part of the question, but it's an interesting

thing to ask. So in this case, we use  $f$  of 2, 1 equals-- well let's see-- so that's 6 plus 6 plus 6 equals 18 units squared of cardboard.

Now, I guess one thing you might notice is we didn't use the second derivative test here. And the question is why-- I mean, we concluded it was a global maximum without ever using the second derivative test. Sorry, global minimum. Pardon me. And the second derivative test is the thing that tells us whether things are minimum or maximum or saddle points.

So why didn't we use it? Well, the answer is the second derivative test tells you whether something is a local minimum or maximum or whether it's a saddle point. So if we had applied the second derivative test at this critical point, what we would have learned is that this critical point is a local minimum.

But being a local minimum isn't enough to guarantee that it's a global minimum. Because we could have on the boundary-- or as  $x$  or  $y$  goes to infinity-- we could have that the function value get smaller and smaller without bound. Now in this case, we showed that the function doesn't do that. It gets larger and larger without bound. And so that meant that that minimum point really is the global minimum. But the second derivative test isn't enough to conclude that something is a global minimum on its own. You really do need that extra analysis that we did.

I'll end there.

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